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Autor: Ostrowski, A. M.

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and therefore, using (13)

$$\int_{\Psi(x_0)}^{\gamma\psi(b_{\nu})} f(x) dx \leq \alpha \int_{\psi(x_0)}^{\psi(b_{\nu})} f(x) dx = \alpha \int_{\psi(x_0)}^{\gamma\psi(b_{\nu})} f(x) dx + \alpha \int_{\gamma\psi(b_{\nu})}^{\psi(b_{\nu})} f(x) dx.$$

But the last right hand integral is, by (14), $\leq c \log \frac{1}{\gamma}$, so that we obtain:

$$(1-\alpha)\int_{\Psi(x_0)}^{\gamma\psi(b_{\mathbf{v}})} f(x) dx \leq \int_{\psi(x_0)}^{\Psi(x_0)} f(x) dx + c \log \frac{1}{\gamma}.$$

The convergence of (2) follows now immediately from $\psi(b_v) \to \infty$.

13. Suppose that we have, on the other hand, for an $a > x_0$:

$$\Psi\left(a\right) \leq \psi\left(a\right)$$
.

Proceeding then as in the proof of the Theorem 1 we have, as from $\psi(b_v) \to \infty$ and the total continuity of $\psi(x)$ follows $b_v \to \infty$, for $b_v \ge a$:

$$\int_{\Psi(a)}^{\Psi(b_{\nu})} f(x) dx \leq \alpha \int_{\psi(a)}^{\psi(b_{\nu})} f(x) dx,$$

and, for $v \to \infty$:

$$\int_{\Psi(a)}^{\infty} f(x) dx \leq \alpha \int_{\Psi(a)}^{\infty} f(x) dx.$$

But here the left hand integral is > 0, the right hand integral is majorized by it and the relation is impossible for $\alpha < 1.3$

III. A NEW METHOD FOR NOT NECESSARILY MONOTONIC f(x)

14. Theorem 4. Assume that $\Psi(x)$ is for $x \geq x_0$ a positive and monotonically increasing differentiable function for which

$$\lim_{x \to \infty} \frac{f(\Psi(x))\Psi'(x)}{f(\psi(x))\psi'(x)}$$

³⁾ Observe that in Ermakof's paper [1] the criteria are given in the following form:

 $[\]sum_{i=0}^{\infty} f(v)$ for a monotonic f(x) is convergent or divergent according as

is < 1 or > 1. In the note [2] Ermakof takes $\Psi(x) \equiv x$ which is no essential specialisation. However, the conditions (5) for convergence and (9) for divergence (with the specialisation $\Psi(x) \equiv x$) are already found in the textbooks, see e.g. [3].

 $\Psi'(x)$ is also monotonically increasing and that we have:

$$\Psi(x) > x \quad (x \ge x_0) \,. \tag{16}$$

Suppose further that f(x) is > 0 for $x \ge x_0$ and integrable and bounded from below by a positive number in any finite subinterval of $\langle x_0, \infty \rangle$. If we have for all $x \ge x_0$:

$$f(\Psi(x)) \Psi'(x) \ge f(x),$$
 (17)

the sum

$$\sum_{v \ge x_0} f(v) \tag{18}$$

is divergent.

15. Proof. Introduce the function

$$F(x) = \inf_{x_0 \le u \le x} f(u); \qquad (19)$$

then F(x) is monotonically decreasing and we have for each $x \ge x_0$:

$$F(x) = \lim_{\kappa \to \infty} f(u_{\kappa})$$

for a convenient sequence u_{κ} from the interval $\langle x_0, x \rangle$.

We can write therefore for a certain sequence v_{κ} from the interval $\langle x_0, x \rangle$:

$$F\left(\Psi(x)\right)\Psi'(x) = \lim_{\kappa \to \infty} f\left(\Psi\left(v_{\kappa}\right)\right)\Psi'(x) \ge \overline{\lim_{\kappa \to \infty}} f\left(\Psi\left(v_{\kappa}\right)\right)\Psi'\left(v_{\kappa}\right).$$

This is, however, by (17) $\geq \overline{\lim} f(v_{\kappa}) \geq F(x)$.

It follows

$$F(\Psi(x)) \Psi'(x) \ge F(x)$$
,

so that the integral $\int_{0}^{\infty} F(x) dx$ is divergent. Since F(x) is monotonic, the same follows for the series $\sum_{0}^{\infty} F(v)$ which has (18) as a majorant. The Theorem 4 is proved.

16. Theorem 5. Assume that $\Psi(x)$ is for $x \geq x_0$ a positive and monotonically increasing differentiable function for which (16) holds. Assume further that $\Psi'(x)$ is either, from a certain x on, monotonically increasing or, for $x \to \infty$, convergent to a finite

limit ω . Assume finally that f(x) is ≥ 0 for $x \geq x_0$, measurable and bounded in each interval $x_0 \leq x \leq a$ and satisfies for all $x \geq x_0$ and for a certain constant $\delta < 1$ the inequality:

$$f(\Psi(x)) \Psi'(x) \le \delta f(x) \quad (x \ge x_0). \tag{20}$$

Then the series (18) is convergent.

17. Proof. Take a number β with $1 > \beta > \delta$. Observe that $\Psi'(x)$ certainly cannot have for $x \to \infty$ a limit $\omega < 1$. For otherwise we would have, with $x \to \infty$,

$$(\Psi(x)-x)' \rightarrow \omega - 1 < 0, \quad \Psi(x) - x \rightarrow -\infty$$

contrary to (16).

We have therefore in any case, from a certain x on, $\Psi'(x) \ge \delta$, and, by (20), $f(\Psi(x)) \le f(x)$. We can therefore assume, changing x_0 if necessary, that we have:

$$f(\Psi(x)) \le f(x) \quad (x \ge x_0).$$
 (21)

Further, if we have $\Psi(x) \to \omega \ge 1$ and if ω is finite there certainly exists an x_1 such that we have, if $x \ge x_1$, $y \ge x_1$,

$$\frac{\delta}{\beta} \leq \frac{\Psi'(x)}{\Psi'(y)} \leq \frac{\beta}{\delta}.$$

We can therefore assume, increasing x_0 if necessary, that we have:

$$\Psi'(x) \le \frac{\beta}{\delta} \Psi'(y) \quad (y \ge x \ge x_0),$$
 (22)

and this is obviously also true if $\Psi'(x)$ is monotonically increasing, so that we can now assume (22) as being true under the conditions of our Theorem.

18. Put

$$x_0 = a_0, \ \Psi(a_0) = a_1, ..., \Psi(a_v) = a_{v+1}, ...$$

The sequence a_{ν} is monotonically increasing. If $\lim a_{\nu} = \tau$ were finite, we would have $\Psi(\tau) = \tau$, contrary to (16). Therefore we have $a_{\nu} \uparrow \infty$.

We have therefore for any $x \ge x_0$ an index v such that $a_v \le x < a_{v+1}$.

Denoting by c an upper bound for f(x) in the interval $\langle a_0, a_1 \rangle$ it follows then from (21):

$$f(x) \le c \quad (x \ge x_0) .$$

19. Put

$$G(x) = \sup_{u \ge x} f(u). \tag{23}$$

G(x) is finite and monotonically decreasing and we have:

$$f(x) \le G(x) \quad (x \ge x_0). \tag{24}$$

By (23), there exists for any $x \ge x_0$ a sequence of numbers u_{κ} , $u_{\kappa} \ge x$ such that $G(\Psi(x)) = \lim_{\kappa \to \infty} f(\Psi(u_{\kappa}))$ and by (22)

$$G\left(\Psi\left(x\right)\right)\Psi'(x) = \lim_{\kappa \to \infty} f\left(\Psi\left(u_{\kappa}\right)\right)\Psi'(x) \leq \overline{\lim_{\kappa \to \infty}} \ f\bigl(\Psi\left(u_{\kappa}\right)\bigr) \frac{\beta}{\delta} \ \Psi'(u_{\kappa}) \, .$$

But this is, by (20),

$$\leq \frac{\beta}{\delta} \delta \overline{\lim}_{\kappa \to \infty} f(u_{\kappa}) \leq \beta G(x).$$

20. We have therefore

$$G(\Psi(x)) \Psi'(x) \leq \beta G(x)$$
,

so that $\int_{0}^{\infty} G(x) dx$ is convergent. But then, since G(x) is monotonically decreasing, the series $\sum_{0}^{\infty} G(v)$ is convergent too, and, by (24), the same holds for the series (18). The Theorem 5 is proved.

21. Theorem 6. Assume that $\Psi(x)$ is for $x \geq x_0$ a positive and monotonically increasing differentiable function for which we have (16). Suppose further that f(x) is > 0 for $x \geq x_0$, is integrable and bounded from below by a positive number in any finite subinterval of $\langle x_0, \infty \rangle$ and satisfies for a constant $\beta > 1$ and for all $x \geq x_0$ the condition

$$f(\Psi(x)) \Psi'(x) \ge \beta f(x), x \ge x_0.$$
 (25)

Finally assume that there exists an $x_1 \ge x_0$ such that we have for all x, u with $x \ge u \ge x_1$:

$$\frac{\Psi'(x)}{\Psi'(u)} \ge \frac{1}{\beta} (x \ge u \ge x_1). \tag{26}$$

Then the series (18) is divergent.

22. Observe that the condition (26) is certainly satisfied from a certain x_1 on, if $\Psi(x)$ has a finite limit ω ,

$$\Psi'(x) \to \omega < \infty (x \to \infty).$$
 (27)

23. Proof of the Theorem 6. Since x_0 can be replaced by any greater number we can assume, without loss of generality, that $x_1 = x_0$. Then we proceed as in the proof of the Theorem 4 defining F(x) by (19) and obtain, as in the section 15, using (26):

$$F\left(\Psi(x)\right)\Psi'(x) = \lim_{\kappa \to \infty} f\left(\Psi(v_{\kappa})\right)\Psi'(x) \ge \frac{1}{\beta} \overline{\lim_{\kappa \to \infty}} f\left(\Psi(v_{\kappa})\right) \Psi'(v_{\kappa})$$
$$\ge \overline{\lim_{\kappa \to \infty}} f\left(v_{\kappa}\right) \ge F\left(x\right).$$

24. We see that F(x) satisfies the conditions of the Theorem 2; therefore the integral $\int_{0}^{\infty} F(x) dx$ is divergent and the same holds for the series $\sum_{i=1}^{\infty} F(v)$, as F(x) is monotonically decreasing. But then the series (18) is also divergent since f(x) is a majorant of F(x). The Theorem 6 is proved.

IV. Another method in the case of divergence

- 25. Theorem 7. The assertion of the Theorem 4 remains valid if the assumption that $\Psi'(x)$ is monotonically increasing is replaced by the assumption that $\Psi'(x)$ is monotonically decreasing.
- 26. *Proof.* Since in any case $\Psi'(x) \ge 0$ there exists a finite ω such that

$$\Psi'(x) \downarrow \omega \quad (x \to \infty)$$

and, as in the sec. 17, we see that this limit is ≥ 1 .