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and therefore, using (13)

$$\int_{\Psi(x_0)}^{\gamma\psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(x_0)}^{\psi(b_\nu)} f(x)\,dx = \alpha \int_{\psi(x_0)}^{\gamma\psi(b_\nu)} f(x)\,dx + \alpha \int_{\gamma\psi(b_\nu)}^{\psi(b_\nu)} f(x)\,dx .$$

But the last right hand integral is, by (14), $\leqq c \log \frac{1}{\gamma}$, so that we obtain:

$$(1-\alpha) \int_{\Psi(x_0)}^{\gamma\psi(b_\nu)} f(x)\,dx \leqq \int_{\psi(x_0)}^{\Psi(x_0)} f(x)\,dx + c \log \frac{1}{\gamma}.$$

The convergence of (2) follows now immediately from $\psi(b_\nu) \to \infty$.

13. Suppose that we have, on the other hand, for an $a > x_0$:

$$\Psi(a) \leqq \psi(a).$$

Proceeding then as in the proof of the Theorem 1 we have, as from $\psi(b_\nu) \to \infty$ and the total continuity of $\psi(x)$ follows $b_\nu \to \infty$, for $b_\nu \geqq a$:

$$\int_{\Psi(a)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(a)}^{\psi(b_\nu)} f(x)\,dx ,$$

and, for $\nu \to \infty$:

$$\int_{\Psi(a)}^{\infty} f(x)\,dx \leqq \alpha \int_{\psi(a)}^{\infty} f(x)\,dx .$$

But here the left hand integral is > 0, the right hand integral is majorized by it and the relation is impossible for $\alpha < 1$. [3])

III. A NEW METHOD FOR NOT NECESSARILY MONOTONIC $f(x)$

14. THEOREM 4. *Assume that $\Psi(x)$ is for $x \geqq x_0$ a positive and monotonically increasing differentiable function for which*

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[3]) Observe that in Ermakof's paper [1] the criteria are given in the following form: $\sum^{\infty} f(\nu)$ for a monotonic $f(x)$ is convergent or divergent according as

$$\lim_{x\to\infty} \frac{f(\Psi(x))\Psi'(x)}{f(\psi(x))\psi'(x)}$$

is < 1 or > 1. In the note [2] Ermakof takes $\Psi(x) \equiv x$ which is no essential specialisation. However, the conditions (5) for convergence and (9) for divergence (with the specialisation $\Psi(x) \equiv x$) are already found in the textbooks, see e.g. [3].

$\Psi'(x)$ is also monotonically increasing and that we have:

$$\Psi(x) > x \quad (x \geqq x_0). \tag{16}$$

Suppose further that $f(x)$ is > 0 for $x \geqq x_0$ and integrable and bounded from below by a positive number in any finite subinterval of $\langle x_0, \infty)$. If we have for all $x \geqq x_0$:

$$f(\Psi(x))\, \Psi'(x) \geqq f(x), \tag{17}$$

the sum

$$\sum_{\nu \geqq x_0} f(\nu) \tag{18}$$

is divergent.

15. *Proof.* Introduce the function

$$F(x) = \operatorname*{Inf}_{x_0 \leqq u \leqq x} f(u); \tag{19}$$

then $F(x)$ is monotonically decreasing and we have for each $x \geqq x_0$:

$$F(x) = \lim_{\kappa \to \infty} f(u_\kappa)$$

for a convenient sequence u_κ from the interval $\langle x_0, x\rangle$.

We can write therefore for a certain sequence v_κ from the interval $\langle x_0, x\rangle$:

$$F(\Psi(x))\, \Psi'(x) = \lim_{\kappa \to \infty} f(\Psi(v_\kappa))\, \Psi'(x) \geqq \overline{\lim_{\kappa \to \infty}}\, f(\Psi(v_\kappa))\, \Psi'(v_\kappa).$$

This is, however, by (17) $\geqq \overline{\lim}\, f(v_\kappa) \geqq F(x)$.

It follows

$$F(\Psi(x))\, \Psi'(x) \geqq F(x),$$

so that the integral $\int^{\infty} F(x)\, dx$ is divergent. Since $F(x)$ is monotonic, the same follows for the series $\sum^{\infty} F(\nu)$ which has (18) as a majorant. The Theorem 4 is proved.

16. THEOREM 5. *Assume that $\Psi(x)$ is for $x \geqq x_0$ a positive and monotonically increasing differentiable function for which (16) holds. Assume further that $\Psi'(x)$ is either, from a certain x on, monotonically increasing or, for $x \to \infty$, convergent to a finite*

limit ω. Assume finally that $f(x)$ is $\geqq 0$ for $x \geqq x_0$, measurable and bounded in each interval $x_0 \leqq x \leqq a$ and satisfies for all $x \geqq x_0$ and for a certain constant $\delta < 1$ the inequality:

$$f(\Psi(x))\,\Psi'(x) \leqq \delta f(x) \quad (x \geqq x_0). \tag{20}$$

Then the series (18) is convergent.

17. *Proof.* Take a number β with $1 > \beta > \delta$. Observe that $\Psi'(x)$ certainly cannot have for $x \to \infty$ a limit $\omega < 1$. For otherwise we would have, with $x \to \infty$,

$$(\Psi(x) - x)' \to \omega - 1 < 0, \quad \Psi(x) - x \to -\infty$$

contrary to (16).

We have therefore in any case, from a certain x on, $\Psi'(x) \geqq \delta$, and, by (20), $f(\Psi(x)) \leqq f(x)$. We can therefore assume, changing x_0 if necessary, that we have:

$$f(\Psi(x)) \leqq f(x) \quad (x \geqq x_0). \tag{21}$$

Further, if we have $\Psi(x) \to \omega \geqq 1$ and if ω is finite there certainly exists an x_1 such that we have, if $x \geqq x_1$, $y \geqq x_1$,

$$\frac{\delta}{\beta} \leqq \frac{\Psi'(x)}{\Psi'(y)} \leqq \frac{\beta}{\delta}.$$

We can therefore assume, increasing x_0 if necessary, that we have:

$$\Psi'(x) \leqq \frac{\beta}{\delta}\,\Psi'(y) \quad (y \geqq x \geqq x_0), \tag{22}$$

and this is obviously also true if $\Psi'(x)$ is monotonically increasing, so that we can now assume (22) as being true under the conditions of our Theorem.

18. Put

$$x_0 = a_0, \quad \Psi(a_0) = a_1, \ldots, \Psi(a_\nu) = a_{\nu+1}, \ldots$$

The sequence a_ν is monotonically increasing. If $\lim a_\nu = \tau$ were finite, we would have $\Psi(\tau) = \tau$, contrary to (16). Therefore we have $a_\nu \uparrow \infty$.

We have therefore for any $x \geqq x_0$ an index ν such that $a_\nu \leqq x < a_{\nu+1}$.

Denoting by c an upper bound for $f(x)$ in the interval $\langle a_0, a_1)$ it follows then from (21):

$$f(x) \leqq c \quad (x \geqq x_0).$$

19. Put

$$G(x) = \operatorname*{Sup}_{u \geqq x} f(u). \tag{23}$$

$G(x)$ is finite and monotonically decreasing and we have:

$$f(x) \leqq G(x) \quad (x \geqq x_0). \tag{24}$$

By (23), there exists for any $x \geqq x_0$ a sequence of numbers u_κ, $u_\kappa \geqq x$ such that $G(\Psi(x)) = \lim\limits_{\kappa \to \infty} f(\Psi(u_\kappa))$ and by (22)

$$G(\Psi(x))\,\Psi'(x) = \lim_{\kappa\to\infty} f(\Psi(u_\kappa))\,\Psi'(x) \leq \overline{\lim_{\kappa\to\infty}}\, f(\Psi(u_\kappa))\frac{\beta}{\delta}\,\Psi'(u_\kappa).$$

But this is, by (20),

$$\leqq \frac{\beta}{\delta}\,\delta\,\overline{\lim_{\kappa\to\infty}} f(u_\kappa) \leqq \beta\, G(x).$$

20. We have therefore

$$G(\Psi(x))\,\Psi'(x) \leqq \beta G(x),$$

so that $\int^{\infty} G(x)\,dx$ is convergent. But then, since $G(x)$ is monotonically decreasing, the series $\sum^{\infty} G(\nu)$ is convergent too, and, by (24), the same holds for the series (18). The Theorem 5 is proved.

21. Theorem 6. *Assume that $\Psi(x)$ is for $x \geqq x_0$ a positive and monotonically increasing differentiable function for which we have (16). Suppose further that $f(x)$ is > 0 for $x \geqq x_0$, is integrable and bounded from below by a positive number in any finite subinterval of $\langle x_0, \infty)$ and satisfies for a constant $\beta > 1$ and for all $x \geqq x_0$ the condition*

$$f(\Psi(x))\,\Psi'(x) \geqq \beta f(x), \; x \geqq x_0. \tag{25}$$

Finally assume that there exists an $x_1 \geqq x_0$ *such that we have for all* x, u *with* $x \geqq u \geqq x_1$:

$$\frac{\Psi'(x)}{\Psi'(u)} \geqq \frac{1}{\beta} \, (x \geqq u \geqq x_1) . \qquad (26)$$

Then the series (18) is divergent.

22. Observe that the condition (26) is certainly satisfied from a certain x_1 on, if $\Psi(x)$ has a finite limit ω,

$$\Psi'(x) \to \omega < \infty \, (x \to \infty) . \qquad (27)$$

23. *Proof of the Theorem 6.* Since x_0 can be replaced by any greater number we can assume, without loss of generality, that $x_1 = x_0$. Then we proceed as in the proof of the Theorem 4 defining $F(x)$ by (19) and obtain, as in the section 15, using (26):

$$F(\Psi(x))\, \Psi'(x) = \lim_{\kappa \to \infty} f(\Psi(v_\kappa))\, \Psi'(x) \geqq \frac{1}{\beta} \overline{\lim_{\kappa \to \infty}} f(\Psi(v_\kappa))\, \Psi'(v_\kappa)$$

$$\geqq \overline{\lim_{\kappa \to \infty}} f(v_\kappa) \geqq F(x) .$$

24. We see that $F(x)$ satisfies the conditions of the Theorem 2; therefore the integral $\int^{\infty} F(x)\, dx$ is divergent and the same holds for the series $\sum^{\infty} F(\nu)$, as $F(x)$ is monotonically decreasing. But then the series (18) is also divergent since $f(x)$ is a majorant of $F(x)$. The Theorem 6 is proved.

IV. Another method in the case of divergence

25. Theorem 7. *The assertion of the Theorem 4 remains valid if the assumption that* $\Psi'(x)$ *is monotonically increasing is replaced by the assumption that* $\Psi'(x)$ *is monotonically decreasing.*

26. *Proof.* Since in any case $\Psi'(x) \geqq 0$ there exists a finite ω such that

$$\Psi'(x) \downarrow \omega \quad (x \to \infty)$$

and, as in the sec. 17, we see that this limit is $\geqq 1$.