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and therefore, using (13)

$$\int_{\Psi(x_0)}^{\gamma\psi(b_v)} f(x) dx \leq \alpha \int_{\psi(x_0)}^{\psi(b_v)} f(x) dx = \alpha \int_{\psi(x_0)}^{\gamma\psi(b_v)} f(x) dx + \alpha \int_{\gamma\psi(b_v)}^{\psi(b_v)} f(x) dx .$$

But the last right hand integral is, by (14), $\leq c \log \frac{1}{\gamma}$, so that we obtain:

$$(1 - \alpha) \int_{\Psi(x_0)}^{\gamma\psi(b_v)} f(x) dx \leq \int_{\psi(x_0)}^{\Psi(x_0)} f(x) dx + c \log \frac{1}{\gamma} .$$

The convergence of (2) follows now immediately from $\psi(b_v) \rightarrow \infty$.

13. Suppose that we have, on the other hand, for an $a > x_0$:

$$\Psi(a) \leq \psi(a) .$$

Proceeding then as in the proof of the Theorem 1 we have, as from $\psi(b_v) \rightarrow \infty$ and the total continuity of $\psi(x)$ follows $b_v \rightarrow \infty$, for $b_v \geq a$:

$$\int_{\Psi(a)}^{\Psi(b_v)} f(x) dx \leq \alpha \int_{\psi(a)}^{\psi(b_v)} f(x) dx ,$$

and, for $v \rightarrow \infty$:

$$\int_{\Psi(a)}^{\infty} f(x) dx \leq \alpha \int_{\psi(a)}^{\infty} f(x) dx .$$

But here the left hand integral is > 0 , the right hand integral is majorized by it and the relation is impossible for $\alpha < 1$.³⁾

III. A NEW METHOD FOR NOT NECESSARILY MONOTONIC $f(x)$

14. THEOREM 4. Assume that $\Psi(x)$ is for $x \geq x_0$ a positive and monotonically increasing differentiable function for which

³⁾ Observe that in Ermakof's paper [1] the criteria are given in the following form:
 $\sum_{\infty} f(v)$ for a monotonic $f(x)$ is convergent or divergent according as

$$\lim_{x \rightarrow \infty} \frac{f(\Psi(x))\Psi'(x)}{f(\psi(x))\psi'(x)}$$

is < 1 or > 1 . In the note [2] Ermakof takes $\Psi(x) \equiv x$ which is no essential specialisation. However, the conditions (5) for convergence and (9) for divergence (with the specialisation $\Psi(x) \equiv x$) are already found in the textbooks, see e.g. [3].

$\Psi'(x)$ is also monotonically increasing and that we have:

$$\Psi(x) > x \quad (x \geq x_0). \quad (16)$$

Suppose further that $f(x)$ is > 0 for $x \geq x_0$ and integrable and bounded from below by a positive number in any finite subinterval of $\langle x_0, \infty \rangle$. If we have for all $x \geq x_0$:

$$f(\Psi(x)) \Psi'(x) \geq f(x), \quad (17)$$

the sum

$$\sum_{v \geq x_0} f(v) \quad (18)$$

is divergent.

15. *Proof.* Introduce the function

$$F(x) = \inf_{x_0 \leq u \leq x} f(u); \quad (19)$$

then $F(x)$ is monotonically decreasing and we have for each $x \geq x_0$:

$$F(x) = \lim_{\kappa \rightarrow \infty} f(u_\kappa)$$

for a convenient sequence u_κ from the interval $\langle x_0, x \rangle$.

We can write therefore for a certain sequence v_κ from the interval $\langle x_0, x \rangle$:

$$F(\Psi(x)) \Psi'(x) = \lim_{\kappa \rightarrow \infty} f(\Psi(v_\kappa)) \Psi'(x) \geq \overline{\lim}_{\kappa \rightarrow \infty} f(\Psi(v_\kappa)) \Psi'(v_\kappa).$$

This is, however, by (17) $\geq \overline{\lim}_{\kappa \rightarrow \infty} f(v_\kappa) \geq F(x)$.

It follows

$$F(\Psi(x)) \Psi'(x) \geq F(x),$$

so that the integral $\int_{x_0}^{\infty} F(x) dx$ is divergent. Since $F(x)$ is monotonic, the same follows for the series $\sum_{\nu} F(\nu)$ which has (18) as a majorant. The Theorem 4 is proved.

16. THEOREM 5. Assume that $\Psi(x)$ is for $x \geq x_0$ a positive and monotonically increasing differentiable function for which (16) holds. Assume further that $\Psi'(x)$ is either, from a certain x on, monotonically increasing or, for $x \rightarrow \infty$, convergent to a finite

limit ω . Assume finally that $f(x)$ is ≥ 0 for $x \geq x_0$, measurable and bounded in each interval $x_0 \leq x \leq a$ and satisfies for all $x \geq x_0$ and for a certain constant $\delta < 1$ the inequality:

$$f(\Psi(x)) \Psi'(x) \leq \delta f(x) \quad (x \geq x_0). \quad (20)$$

Then the series (18) is convergent.

17. *Proof.* Take a number β with $1 > \beta > \delta$. Observe that $\Psi'(x)$ certainly cannot have for $x \rightarrow \infty$ a limit $\omega < 1$. For otherwise we would have, with $x \rightarrow \infty$,

$$(\Psi(x) - x)' \rightarrow \omega - 1 < 0, \quad \Psi(x) - x \rightarrow -\infty$$

contrary to (16).

We have therefore in any case, from a certain x on, $\Psi'(x) \geq \delta$, and, by (20), $f(\Psi(x)) \leq f(x)$. We can therefore assume, changing x_0 if necessary, that we have:

$$f(\Psi(x)) \leq f(x) \quad (x \geq x_0). \quad (21)$$

Further, if we have $\Psi(x) \rightarrow \omega \geq 1$ and if ω is finite there certainly exists an x_1 such that we have, if $x \geq x_1, y \geq x_1$,

$$\frac{\delta}{\beta} \leq \frac{\Psi'(x)}{\Psi'(y)} \leq \frac{\beta}{\delta}.$$

We can therefore assume, increasing x_0 if necessary, that we have:

$$\Psi'(x) \leq \frac{\beta}{\delta} \Psi'(y) \quad (y \geq x \geq x_0), \quad (22)$$

and this is obviously also true if $\Psi'(x)$ is monotonically increasing, so that we can now assume (22) as being true under the conditions of our Theorem.

18. Put

$$x_0 = a_0, \quad \Psi(a_0) = a_1, \dots, \Psi(a_v) = a_{v+1}, \dots$$

The sequence a_v is monotonically increasing. If $\lim a_v = \tau$ were finite, we would have $\Psi(\tau) = \tau$, contrary to (16). Therefore we have $a_v \uparrow \infty$.

We have therefore for any $x \geq x_0$ an index ν such that $a_\nu \leq x < a_{\nu+1}$.

Denoting by c an upper bound for $f(x)$ in the interval $\langle a_0, a_1 \rangle$ it follows then from (21):

$$f(x) \leq c \quad (x \geq x_0).$$

19. Put

$$G(x) = \sup_{u \geq x} f(u). \quad (23)$$

$G(x)$ is finite and monotonically decreasing and we have:

$$f(x) \leq G(x) \quad (x \geq x_0). \quad (24)$$

By (23), there exists for any $x \geq x_0$ a sequence of numbers u_κ , $u_\kappa \geq x$ such that $G(\Psi(x)) = \lim_{\kappa \rightarrow \infty} f(\Psi(u_\kappa))$ and by (22)

$$G(\Psi(x)) \Psi'(x) = \lim_{\kappa \rightarrow \infty} f(\Psi(u_\kappa)) \Psi'(x) \leq \overline{\lim}_{\kappa \rightarrow \infty} f(\Psi(u_\kappa)) \frac{\beta}{\delta} \Psi'(u_\kappa).$$

But this is, by (20),

$$\leq \frac{\beta}{\delta} \delta \overline{\lim}_{\kappa \rightarrow \infty} f(u_\kappa) \leq \beta G(x).$$

20. We have therefore

$$G(\Psi(x)) \Psi'(x) \leq \beta G(x),$$

so that $\int^\infty G(x) dx$ is convergent. But then, since $G(x)$ is monotonically decreasing, the series $\sum_{\nu} G(\nu)$ is convergent too, and, by (24), the same holds for the series (18). The Theorem 5 is proved.

21. THEOREM 6. Assume that $\Psi(x)$ is for $x \geq x_0$ a positive and monotonically increasing differentiable function for which we have (16). Suppose further that $f(x)$ is > 0 for $x \geq x_0$, is integrable and bounded from below by a positive number in any finite subinterval of $\langle x_0, \infty \rangle$ and satisfies for a constant $\beta > 1$ and for all $x \geq x_0$ the condition

$$f(\Psi(x)) \Psi'(x) \geq \beta f(x), \quad x \geq x_0. \quad (25)$$

Finally assume that there exists an $x_1 \geq x_0$ such that we have for all x, u with $x \geq u \geq x_1$:

$$\frac{\Psi'(x)}{\Psi'(u)} \geq \frac{1}{\beta} \quad (x \geq u \geq x_1). \quad (26)$$

Then the series (18) is divergent.

22. Observe that the condition (26) is certainly satisfied from a certain x_1 on, if $\Psi(x)$ has a finite limit ω ,

$$\Psi'(x) \rightarrow \omega < \infty \quad (x \rightarrow \infty). \quad (27)$$

23. *Proof of the Theorem 6.* Since x_0 can be replaced by any greater number we can assume, without loss of generality, that $x_1 = x_0$. Then we proceed as in the proof of the Theorem 4 defining $F(x)$ by (19) and obtain, as in the section 15, using (26):

$$\begin{aligned} F(\Psi(x)) \Psi'(x) &= \lim_{\kappa \rightarrow \infty} f(\Psi(v_\kappa)) \Psi'(x) \geq \frac{1}{\beta} \overline{\lim}_{\kappa \rightarrow \infty} f(\Psi(v_\kappa)) \Psi'(v_\kappa) \\ &\geq \overline{\lim}_{\kappa \rightarrow \infty} f(v_\kappa) \geq F(x). \end{aligned}$$

24. We see that $F(x)$ satisfies the conditions of the Theorem 2; therefore the integral $\int_0^\infty F(x) dx$ is divergent and the same holds for the series $\sum_0^\infty F(\rho)$, as $F(x)$ is monotonically decreasing. But then the series (18) is also divergent since $f(x)$ is a majorant of $F(x)$. The Theorem 6 is proved.

IV. ANOTHER METHOD IN THE CASE OF DIVERGENCE

25. **THEOREM 7.** *The assertion of the Theorem 4 remains valid if the assumption that $\Psi'(x)$ is monotonically increasing is replaced by the assumption that $\Psi'(x)$ is monotonically decreasing.*

26. *Proof.* Since in any case $\Psi'(x) \geq 0$ there exists a finite ω such that

$$\Psi'(x) \downarrow \omega \quad (x \rightarrow \infty)$$

and, as in the sec. 17, we see that this limit is ≥ 1 .