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of the problem and prove generalized versions of Pringsheim's results.

This note brings therefore an improvement and simplification of the sections I-V and XI of [5], while I have nothing to add to the sections VI-X of [5].

II. ERMAKOF'S DIRECT METHOD

6. The form of the expression (1) makes it plausible that we will have to use the integral transformation formula

$$\int_a^b f(\Psi(x)) \Psi'(x) dx = \int_{\Psi(a)}^{\Psi(b)} f(x) dx. \quad (3)$$

In order to be able to use (3) we have in any case to assume that $f(x)$ is integrable in the integration interval and $\Psi(x)$ totally continuous between a and b . However, additional conditions are necessary and two such conditions are known either of which ensures the relation (3):

- J_1 : $|f(x)|$ is uniformly bounded in the integration interval;
 J_2 : $\Psi(x)$ is monotonically increasing or monotonically decreasing.

7. THEOREM 1. Assume that $\psi(x)$ and $\Psi(x)$ are totally continuous for $x \geq x_0$ and that we have for a sequence $b_\nu \geq x_0$ ($\nu = 1, 2, \dots$)

$$\psi(b_\nu) \leq \Psi(b_\nu), \quad \Psi(b_\nu) \rightarrow \infty \quad (\nu \rightarrow \infty). \quad (4)$$

Let $f(x)$ be ≥ 0 on no half-line $x \geq \xi$ almost everywhere $= 0$, and measurable in an interval J containing all values of $\psi(x)$ and $\Psi(x)$ for $x \geq x_0$. Assume further that for any finite subinterval of J the transformation formula (3) holds as well for $\psi(x)$ as for $\Psi(x)$. Then, if we have for almost all x with $x \geq x_0$ and for an α with $0 < \alpha < 1$:

$$f(\Psi(x)) \Psi'(x) \leq \alpha f(\psi(x)) \psi'(x) \quad (x \geq x_0), \quad 0 < \alpha < 1, \quad (5)$$

the integral (2) is convergent and we have for all $x \geq x_0$:

$$\Psi(x) > \psi(x) \quad (x \geq x_0). \quad (6)$$

8. *Proof.* For an arbitrary $x \geq x_0$ integrate (5) between x and $b_v > x$. Then we have, using (3):

$$\int_{\Psi(x)}^{\Psi(b_v)} f(x) dx \leq \alpha \int_{\psi(x)}^{\psi(b_v)} f(x) dx$$

and this remains true, by (4), if $\psi(b_v)$ is replaced by $\Psi(b_v)$. We can therefore write

$$\int_{\Psi(x)}^{\Psi(b_v)} f(x) dx \leq \alpha \int_{\Psi(x)}^{\Psi(b_v)} f(x) dx + \alpha \int_{\psi(x)}^{\Psi(x)} f(x) dx,$$

or, bringing the first right hand term to the left:

$$(1 - \alpha) \int_{\Psi(x)}^{\Psi(b_v)} f(x) dx \leq \alpha \int_{\psi(x)}^{\Psi(x)} f(x) dx.$$

But here, if we take $x = b_1$ it follows for $b_v \rightarrow \infty$ the convergence of (2) and also that the right hand expression is > 0 for any $x \geq x_0$. (6) follows immediately and the Theorem 1 is proved.

9. **THEOREM 2.** Assume that $\psi(x)$, $\Psi(x)$ are totally continuous for $x \geq x_0$ and that $f(x)$ is non-negative and measurable in an interval J containing all values of $\psi(x)$ and $\Psi(x)$. Assume that (3) holds as well for $\psi(x)$ as for $\Psi(x)$. Assume further that there exists an $a \geq x_0$ such that:

$$\int_{\psi(a)}^{\Psi(a)} f(x) dx > 0, \quad (7)$$

and a sequence $b_v \geq x_0$ ($v = 1, 2, \dots$) such that:

$$\psi(b_v) \rightarrow \infty, \quad \Psi(b_v) \rightarrow \infty \quad (v \rightarrow \infty). \quad (8)$$

Then if we have for almost all $x \geq x_0$:

$$f(\Psi(x)) \Psi'(x) \geq f(\psi(x)) \psi'(x), \quad (9)$$

the integral (2) is divergent and we have for all $x \geq a$:

$$\Psi(x) > \psi(x) \quad (x \geq a). \quad (10)$$

10. *Proof.* For any $x > a$ we obtain from (9), integrating on both sides from a to x and using (3):

$$\int_{\Psi(a)}^{\Psi(x)} f(x) dx \geq \int_{\psi(a)}^{\psi(x)} f(x) dx$$

and therefore

$$\int_{\psi(x)}^{\Psi(x)} f(x) dx \geq \int_{\psi(a)}^{\Psi(a)} f(x) dx \quad (x \geq a). \quad (11)$$

This proves already (10).

Putting in (11) $x = b_v$ it follows

$$\int_{\psi(b_v)}^{\Psi(b_v)} f(x) dx \geq \int_{\psi(a)}^{\Psi(a)} f(x) dx \quad (12)$$

while, if (2) were convergent, the left side integral in (12) would tend to 0.

Theorem 2 is proved.

11. **THEOREM 3.** Assume that $\psi(x)$ and $\Psi(x)$ are totally continuous for $x \geq x_0$, that (3) holds as well for $\psi(x)$ as for $\Psi(x)$ and $f(x)$ is ≥ 0 and measurable in an interval containing all values of $\psi(x)$ and $\Psi(x)$ for $x > x_0$ without being almost everywhere $= 0$ in $(\Psi(a), \infty)$. Assume further that there exists a constant γ , $0 < \gamma < 1$, and a sequence $b_v \geq x_0$ ($v = 1, 2, \dots$) such that

$$\gamma \psi(b_v) \leq \Psi(b_v), \quad 0 < \gamma < 1, \quad \psi(b_v) \rightarrow \infty \quad (v \rightarrow \infty), \quad (13)$$

and further that for a constant c from a certain $x = x_1 \geq x_0$ on:

$$f(x) \leq \frac{c}{x} \quad (x \geq x_1). \quad (14)$$

Assume finally that for a constant α , $0 < \alpha < 1$:

$$f(\Psi(x)) \Psi'(x) \leq \alpha f(\psi(x)) \psi'(x), \quad 0 < \alpha < 1. \quad (15)$$

Then the integral (2) converges and we have $\Psi(x) > \psi(x)$ for all $x > x_0$.

12. *Proof.* We have as in the proof of the Theorem 1:

$$\int_{\Psi(x_0)}^{\Psi(b_v)} f(x) dx \leq \alpha \int_{\psi(x_0)}^{\psi(b_v)} f(x) dx$$

and therefore, using (13)

$$\int_{\Psi(x_0)}^{\gamma\psi(b_v)} f(x) dx \leq \alpha \int_{\psi(x_0)}^{\psi(b_v)} f(x) dx = \alpha \int_{\psi(x_0)}^{\gamma\psi(b_v)} f(x) dx + \alpha \int_{\gamma\psi(b_v)}^{\psi(b_v)} f(x) dx .$$

But the last right hand integral is, by (14), $\leq c \log \frac{1}{\gamma}$, so that we obtain:

$$(1 - \alpha) \int_{\Psi(x_0)}^{\gamma\psi(b_v)} f(x) dx \leq \int_{\psi(x_0)}^{\Psi(x_0)} f(x) dx + c \log \frac{1}{\gamma} .$$

The convergence of (2) follows now immediately from $\psi(b_v) \rightarrow \infty$.

13. Suppose that we have, on the other hand, for an $a > x_0$:

$$\Psi(a) \leq \psi(a) .$$

Proceeding then as in the proof of the Theorem 1 we have, as from $\psi(b_v) \rightarrow \infty$ and the total continuity of $\psi(x)$ follows $b_v \rightarrow \infty$, for $b_v \geq a$:

$$\int_{\Psi(a)}^{\Psi(b_v)} f(x) dx \leq \alpha \int_{\psi(a)}^{\psi(b_v)} f(x) dx ,$$

and, for $v \rightarrow \infty$:

$$\int_{\Psi(a)}^{\infty} f(x) dx \leq \alpha \int_{\psi(a)}^{\infty} f(x) dx .$$

But here the left hand integral is > 0 , the right hand integral is majorized by it and the relation is impossible for $\alpha < 1$.³⁾

III. A NEW METHOD FOR NOT NECESSARILY MONOTONIC $f(x)$

14. THEOREM 4. Assume that $\Psi(x)$ is for $x \geq x_0$ a positive and monotonically increasing differentiable function for which

³⁾ Observe that in Ermakof's paper [1] the criteria are given in the following form:
 $\sum_{\infty} f(v)$ for a monotonic $f(x)$ is convergent or divergent according as

$$\lim_{x \rightarrow \infty} \frac{f(\Psi(x))\Psi'(x)}{f(\psi(x))\psi'(x)}$$

is < 1 or > 1 . In the note [2] Ermakof takes $\Psi(x) \equiv x$ which is no essential specialisation. However, the conditions (5) for convergence and (9) for divergence (with the specialisation $\Psi(x) \equiv x$) are already found in the textbooks, see e.g. [3].