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Artikel: ON ERMAKOF'S CONVERGENCE CRITERIA AND ABEL'S FUNCTIONAL EQUATION.

Autor: Ostrowski, A. M.

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of the problem and prove generalized versions of Pringsheim's results.

This note brings therefore an improvement and simplification of the sections I-V and XI of [5], while I have nothing to add to the sections VI-X of [5].

II. Ermakof's direct Method

6. The form of the expression (1) makes it plausible that we will have to use the integral transformation formula

$$\int_a^b f(\Psi(x))\,\Psi'(x)\,dx = \int_{\Psi(a)}^{\Psi(b)} f(x)\,dx. \tag{3}$$

In order to be able to use (3) we have in any case to assume that $f(x)$ is integrable in the integration interval and $\Psi(x)$ totally continuous between a and b. However, additional conditions are necessary and two such conditions are known either of which ensures the relation (3):

J_1: $|f(x)|$ is uniformly bounded in the integration interval;
J_2: $\Psi(x)$ is monotonically increasing or monotonically decreasing.

7. Theorem 1. *Assume that $\psi(x)$ and $\Psi(x)$ are totally continuous for $x \geqq x_0$ and that we have for a sequence $b_\nu \geqq x_0$ $(\nu = 1,2, \ldots)$*

$$\psi(b_\nu) \leqq \Psi(b_\nu),\ \Psi(b_\nu) \to \infty\ (\nu \to \infty). \tag{4}$$

Let $f(x)$ be $\geqq 0$ on no half-line $x \geqq \xi$ almost everywhere $= 0$, and measurable in an interval J containing all values of $\psi(x)$ and $\Psi(x)$ for $x \geqq x_0$. Assume further that for any finite subinterval of J the transformation formula (3) *holds as well for $\psi(x)$ as for $\Psi(x)$. Then, if we have for almost all x with $x \geqq x_0$ and for an α with $0 < \alpha < 1$:*

$$f(\Psi(x))\,\Psi'(x) \leqq \alpha f(\psi(x))\,\psi'(x)\ (x \geqq x_0),\ 0 < \alpha < 1, \tag{5}$$

the integral (2) *is convergent and we have for all $x \geqq x_0$:*

$$\Psi(x) > \psi(x) \quad (x \geqq x_0). \tag{6}$$

8. *Proof.* For an arbitrary $x \geqq x_0$ integrate (5) between x and $b_\nu > x$. Then we have, using (3):

$$\int_{\Psi(x)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(x)}^{\psi(b_\nu)} f(x)\,dx$$

and this remains true, by (4), if $\psi(b_\nu)$ is replaced by $\Psi(b_\nu)$. We can therefore write

$$\int_{\Psi(x)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\Psi(x)}^{\Psi(b_\nu)} f(x)\,dx + \alpha \int_{\psi(x)}^{\Psi(x)} f(x)\,dx\,,$$

or, bringing the first right hand term to the left:

$$(1-\alpha) \int_{\Psi(x)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(x)}^{\Psi(x)} f(x)\,dx\,.$$

But here, if we take $x = b_1$ it follows for $b_\nu \to \infty$ the convergence of (2) and also that the right hand expression is > 0 for any $x \geqq x_0$. (6) follows immediately and the Theorem 1 is proved.

9. Theorem 2. *Assume that $\psi(x)$, $\Psi(x)$ are totally continuous for $x \geqq x_0$ and that $f(x)$ is non-negative and measurable in an interval J containing all values of $\psi(x)$ and $\Psi(x)$. Assume that (3) holds as well for $\psi(x)$ as for $\Psi(x)$. Assume further that there exists an $a \geqq x_0$ such that:*

$$\int_{\psi(a)}^{\Psi(a)} f(x)\,dx > 0\,, \tag{7}$$

and a sequence $b_\nu \geqq x_0$ $(\nu = 1,2\ldots)$ such that:

$$\psi(b_\nu) \to \infty\,, \quad \Psi(b_\nu) \to \infty \; (\nu \to \infty)\,. \tag{8}$$

Then if we have for almost all $x \geqq x_0$:

$$f(\Psi(x))\,\Psi'(x) \geqq f(\psi(x))\,\psi'(x)\,, \tag{9}$$

the integral (2) is divergent and we have for all $x \geqq a$:

$$\Psi(x) > \psi(x) \quad (x \geqq a)\,. \tag{10}$$

10. *Proof*. For any $x > a$ we obtain from (9), integrating on both sides from a to x and using (3):

$$\int_{\Psi(a)}^{\Psi(x)} f(x)\,dx \geqq \int_{\psi(a)}^{\psi(x)} f(x)\,dx$$

and therefore

$$\int_{\psi(x)}^{\Psi(x)} f(x)\,dx \geqq \int_{\psi(a)}^{\Psi(a)} f(x)\,dx \quad (x \geqq a)\,. \tag{11}$$

This proves already (10).

Putting in (11) $x = b_\nu$ it follows

$$\int_{\psi(b_\nu)}^{\Psi(b_\nu)} f(x)\,dx \geqq \int_{\psi(a)}^{\Psi(a)} f(x)\,dx \tag{12}$$

while, if (2) were convergent, the left side integral in (12) would tend to 0.

Theorem 2 is proved.

11. Theorem 3. *Assume that $\psi(x)$ and $\Psi(x)$ are totally continuous for $x \geqq x_0$, that (3) holds as well for $\psi(x)$ as for $\Psi(x)$ and $f(x)$ is $\geqq 0$ and measurable in an interval containing all values of $\psi(x)$ and $\Psi(x)$ for $x > x_0$ without being almost everywhere $= 0$ in $(\Psi(a), \infty)$. Assume further that there exists a constant γ, $0 < \gamma < 1$, and a sequence $b_\nu \geqq x_0$ $(\nu = 1,2,\ldots)$ such that*

$$\gamma\psi(b_\nu) \leqq \Psi(b_\nu)\,,\ 0 < \gamma < 1\,,\ \psi(b_\nu) \to \infty\ (\nu \to \infty)\,, \tag{13}$$

and further that for a constant c from a certain $x = x_1 \geqq x_0$ on:

$$f(x) \leqq \frac{c}{x} \quad (x \geqq x_1)\,. \tag{14}$$

Assume finally that for a constant α, $0 < \alpha < 1$:

$$f(\Psi(x))\,\Psi'(x) \leqq \alpha f(\psi(x))\,\psi'(x), \quad 0 < \alpha < 1\,. \tag{15}$$

Then the integral (2) converges and we have $\Psi(x) > \psi(x)$ for all $x > x_0$.

12. *Proof*. We have as in the proof of the Theorem 1:

$$\int_{\Psi(x_0)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(x_0)}^{\psi(b_\nu)} f(x)\,dx$$

and therefore, using (13)

$$\int_{\Psi(x_0)}^{\gamma\psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(x_0)}^{\psi(b_\nu)} f(x)\,dx = \alpha \int_{\psi(x_0)}^{\gamma\psi(b_\nu)} f(x)\,dx + \alpha \int_{\gamma\psi(b_\nu)}^{\psi(b_\nu)} f(x)\,dx\,.$$

But the last right hand integral is, by (14), $\leqq c \log\frac{1}{\gamma}$, so that we obtain:

$$(1-\alpha) \int_{\Psi(x_0)}^{\gamma\psi(b_\nu)} f(x)\,dx \leqq \int_{\psi(x_0)}^{\Psi(x_0)} f(x)\,dx + c\,\log\frac{1}{\gamma}\,.$$

The convergence of (2) follows now immediately from $\psi(b_\nu) \to \infty$.

13. Suppose that we have, on the other hand, for an $a > x_0$:

$$\Psi(a) \leqq \psi(a)\,.$$

Proceeding then as in the proof of the Theorem 1 we have, as from $\psi(b_\nu) \to \infty$ and the total continuity of $\psi(x)$ follows $b_\nu \to \infty$, for $b_\nu \geqq a$:

$$\int_{\Psi(a)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(a)}^{\psi(b_\nu)} f(x)\,dx\,,$$

and, for $\nu \to \infty$:

$$\int_{\Psi(a)}^{\infty} f(x)\,dx \leqq \alpha \int_{\psi(a)}^{\infty} f(x)\,dx\,.$$

But here the left hand integral is > 0, the right hand integral is majorized by it and the relation is impossible for $\alpha < 1$. [3])

III. A new method for not necessarily monotonic $f(x)$

14. Theorem 4. *Assume that $\Psi(x)$ is for $x \geqq x_0$ a positive and monotonically increasing differentiable function for which*

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[3]) Observe that in Ermakof's paper [1] the criteria are given in the following form: $\sum^{\infty} f(\nu)$ for a monotonic $f(x)$ is convergent or divergent according as

$$\lim_{x\to\infty} \frac{f(\Psi(x))\Psi'(x)}{f(\psi(x))\psi'(x)}$$

is < 1 or > 1. In the note [2] Ermakof takes $\Psi(x) \equiv x$ which is no essential specialisation. However, the conditions (5) for convergence and (9) for divergence (with the specialisation $\Psi(x) \equiv x$) are already found in the textbooks, see e.g. [3].