Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 11 (1965)

Heft: 2-3: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON ERMAKOF'S CONVERGENCE CRITERIA AND ABEL'S

FUNCTIONAL EQUATION.

Autor: Ostrowski, A. M.

Kapitel: II. Ermakof's direct Method

DOI: https://doi.org/10.5169/seals-39968

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 28.11.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

of the problem and prove generalized versions of Pringsheim's results.

This note brings therefore an improvement and simplification of the sections I-V and XI of [5], while I have nothing to add to the sections VI-X of [5].

II. Ermakof's direct Method

6. The form of the expression (1) makes it plausible that we will have to use the integral transformation formula

$$\int_{a}^{b} f(\Psi(x)) \Psi'(x) dx = \int_{\Psi(a)}^{\Psi(b)} f(x) dx.$$
 (3)

In order to be able to use (3) we have in any case to assume that f(x) is integrable in the integration interval and $\Psi(x)$ totally continuous between a and b. However, additional conditions are necessary and two such conditions are known either of which ensures the relation (3):

 J_1 : |f(x)| is uniformly bounded in the integration interval; J_2 : $\Psi(x)$ is monotonically increasing or monotonically decreasing.

7. Theorem 1. Assume that ψ (x) and Ψ (x) are totally continuous for $x \geq x_0$ and that we have for a sequence $b_{\mathbf{v}} \geq x_0$ ($\varphi = 1, 2, ...$)

$$\psi(b_{\nu}) \leq \Psi(b_{\nu}), \Psi(b_{\nu}) \to \infty (\nu \to \infty).$$
(4)

Let f(x) be ≥ 0 on no half-line $x \geq \xi$ almost everywhere = 0, and measurable in an interval J containing all values of $\psi(x)$ and $\Psi(x)$ for $x \geq x_0$. Assume further that for any finite subinterval of J the transformation formula (3) holds as well for $\psi(x)$ as for $\Psi(x)$. Then, if we have for almost all x with $x \geq x_0$ and for an α with $0 < \alpha < 1$:

$$f(\Psi(x)) \Psi'(x) \le \alpha f(\psi(x)) \psi'(x) (x \ge x_0), \ 0 < \alpha < 1, \tag{5}$$

the integral (2) is convergent and we have for all $x \ge x_0$:

$$\Psi(x) > \psi(x) \quad (x \ge x_0). \tag{6}$$

8. Proof. For an arbitrary $x \ge x_0$ integrate (5) between x and $b_v > x$. Then we have, using (3):

$$\int_{\Psi(x)}^{\Psi(b_{\nu})} f(x) dx \leq \alpha \int_{\psi(x)}^{\psi(b_{\nu})} f(x) dx$$

and this remains true, by (4), if ψ (b_{ν}) is replaced by Ψ (b_{ν}). We can therefore write

$$\int_{\Psi(x)}^{\Psi(b_{\nu})} f(x) dx \leq \alpha \int_{\Psi(x)}^{\Psi(b_{\nu})} f(x) dx + \alpha \int_{\psi(x)}^{\Psi(x)} f(x) dx,$$

or, bringing the first right hand term to the left:

$$(1-\alpha)\int_{\Psi(x)}^{\Psi(b_{\mathbf{v}})} f(x) dx \leq \alpha \int_{\psi(x)}^{\Psi(x)} f(x) dx.$$

But here, if we take $x = b_1$ it follows for $b_{\nu} \to \infty$ the convergence of (2) and also that the right hand expression is > 0 for any $x \ge x_0$. (6) follows immediately and the Theorem 1 is proved.

9. Theorem 2. Assume that $\psi(x)$, $\Psi(x)$ are totally continuous for $x \geq x_0$ and that f(x) is non-negative and measurable in an interval J containing all values of $\psi(x)$ and $\Psi(x)$. Assume that (3) holds as well for $\psi(x)$ as for $\Psi(x)$. Assume further that there exists an $a \geq x_0$ such that:

$$\int_{\psi(a)}^{\Psi(a)} f(x) dx > 0, \qquad (7)$$

and a sequence $b_{\mathbf{v}} \geq x_{\mathbf{0}}$ ($\mathbf{v} = 1,2...$) such that:

$$\psi(b_{\nu}) \to \infty$$
, $\Psi(b_{\nu}) \to \infty (\nu \to \infty)$. (8)

Then if we have for almost all $x \ge x_0$:

$$f(\Psi(x)) \Psi'(x) \ge f(\psi(x)) \psi'(x), \qquad (9)$$

the integral (2) is divergent and we have for all $x \ge a$:

$$\Psi(x) > \psi(x) \quad (x \ge a). \tag{10}$$

10. Proof. For any x > a we obtain from (9), integrating on both sides from a to x and using (3):

$$\int_{\Psi(a)}^{\Psi(x)} f(x) dx \ge \int_{\psi(a)}^{\psi(x)} f(x) dx$$

and therefore

$$\int_{\psi(x)}^{\Psi(x)} f(x) dx \ge \int_{\psi(a)}^{\Psi(a)} f(x) dx \quad (x \ge a).$$
 (11)

This proves already (10).

Putting in (11) $x = b_{\nu}$ it follows

$$\int_{\psi(b_{\nu})}^{\Psi(b_{\nu})} f(x) dx \ge \int_{\psi(a)}^{\Psi(a)} f(x) dx \tag{12}$$

while, if (2) were convergent, the left side integral in (12) would tend to 0.

Theorem 2 is proved.

11. Theorem 3. Assume that $\psi(x)$ and $\Psi(x)$ are totally continuous for $x \geq x_0$, that (3) holds as well for $\psi(x)$ as for $\Psi(x)$ and f(x) is ≥ 0 and measurable in an interval containing all values of $\psi(x)$ and $\Psi(x)$ for $x > x_0$ without being almost everywhere = 0 in $(\Psi(a), \infty)$. Assume further that there exists a constant γ , $0 < \gamma < 1$, and a sequence $b_{\gamma} \geq x_0$ $(\gamma = 1, 2, ...)$ such that

$$\gamma \psi (b_{\nu}) \leq \Psi (b_{\nu}), \ 0 < \gamma < 1, \ \psi (b_{\nu}) \rightarrow \infty (\nu \rightarrow \infty), \ (13)$$

and further that for a constant c from a certain $x = x_1 \ge x_0$ on:

$$f(x) \le \frac{c}{x} \quad (x \ge x_1) \ . \tag{14}$$

Assume finally that for a constant α , $0 < \alpha < 1$:

$$f(\Psi(x))\Psi'(x) \le \alpha f(\psi(x))\psi'(x), \quad 0 < \alpha < 1.$$
 (15)

Then the integral (2) converges and we have $\Psi(x) > \psi(x)$ for all $x > x_0$.

12. Proof. We have as in the proof of the Theorem 1:

$$\int_{\Psi(x_0)}^{\Psi(b_{\nu})} f(x) dx \le \alpha \int_{\psi(x_0)}^{\psi(b_{\nu})} f(x) dx$$

and therefore, using (13)

$$\int_{\Psi(x_0)}^{\gamma\psi(b_{\nu})} f(x) dx \leq \alpha \int_{\psi(x_0)}^{\psi(b_{\nu})} f(x) dx = \alpha \int_{\psi(x_0)}^{\gamma\psi(b_{\nu})} f(x) dx + \alpha \int_{\gamma\psi(b_{\nu})}^{\psi(b_{\nu})} f(x) dx.$$

But the last right hand integral is, by (14), $\leq c \log \frac{1}{\gamma}$, so that we obtain:

$$(1-\alpha)\int_{\Psi(x_0)}^{\gamma\psi(b_{\mathbf{v}})} f(x) dx \leq \int_{\psi(x_0)}^{\Psi(x_0)} f(x) dx + c \log \frac{1}{\gamma}.$$

The convergence of (2) follows now immediately from $\psi(b_v) \to \infty$.

13. Suppose that we have, on the other hand, for an $a > x_0$:

$$\Psi\left(a\right) \leq \psi\left(a\right)$$
.

Proceeding then as in the proof of the Theorem 1 we have, as from $\psi(b_v) \to \infty$ and the total continuity of $\psi(x)$ follows $b_v \to \infty$, for $b_v \ge a$:

$$\int_{\Psi(a)}^{\Psi(b_{\nu})} f(x) dx \leq \alpha \int_{\psi(a)}^{\psi(b_{\nu})} f(x) dx,$$

and, for $v \to \infty$:

$$\int_{\Psi(a)}^{\infty} f(x) dx \leq \alpha \int_{\Psi(a)}^{\infty} f(x) dx.$$

But here the left hand integral is > 0, the right hand integral is majorized by it and the relation is impossible for $\alpha < 1.3$

III. A NEW METHOD FOR NOT NECESSARILY MONOTONIC f(x)

14. Theorem 4. Assume that $\Psi(x)$ is for $x \geq x_0$ a positive and monotonically increasing differentiable function for which

$$\lim_{x \to \infty} \frac{f(\Psi(x))\Psi'(x)}{f(\psi(x))\psi'(x)}$$

³⁾ Observe that in Ermakof's paper [1] the criteria are given in the following form:

 $[\]sum_{i=0}^{\infty} f(v)$ for a monotonic f(x) is convergent or divergent according as

is < 1 or > 1. In the note [2] Ermakof takes $\Psi(x) \equiv x$ which is no essential specialisation. However, the conditions (5) for convergence and (9) for divergence (with the specialisation $\Psi(x) \equiv x$) are already found in the textbooks, see e.g. [3].