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VERTEX POINTS OF FUNCTIONS

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For f a real function of n variables, usually the Hessian matrix is studied in connection with Gaussian and mean curvatures of $f(x_1, \ldots, x_n)$. In this paper we study other properties of f in a neighborhood of a point. In particular we get a method for obtaining vertex points of the function f. We also generalize the idea to some complex cases.

1. Definitions and notations

Let f a function of complex variables $x_1, \ldots, x_n$ be of class C'' in $x_1, \ldots, x_n$, and $\bar{x}_1, \ldots, \bar{x}_n$, in a neighborhood of a point. Then f is called unitarily analytic if

$$\frac{\partial^2 f}{\partial x_i \, \partial \bar{x}_j} = \left(\overline{\frac{\partial^2 f}{\partial \bar{x}_i \, \partial x_j}} \right).$$

Theorem: Let f be of class C'' in $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n$ in a neighborhood of a point, and

$$\frac{\partial f}{\partial \bar{x}_k} = \left(\overline{\frac{\partial f}{\partial x_k}} \right).$$

Then f is unitarily analytic.

The proof is quite simple and we omit it. Note that the converse is not necessarily true.

2. Tangent Quadric

Let f be unitarily analytic in a neighborhood of $(c_1, \ldots, c_n)$.

Let, for example, $\dfrac{\partial f}{\partial c_1}$ be the value of $\dfrac{\partial f}{\partial x_1}$ at $(c_1, \ldots, c_n)$, and

$f_c = f(c_1, ..., c_n)$. Then

$$(x_1 - c_1 \dots x_n - c_n) \begin{bmatrix} \frac{\partial^2 f}{\partial c_1 \, \partial \bar{c}_1} & \cdots & \frac{\partial^2 f}{\partial c_1 \, \partial \bar{c}_n} & \left(\overline{\frac{\partial f}{\partial c_1}}\right) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial^2 f}{\partial c_n \, \partial \bar{c}_1} & & \frac{\partial^2 f}{\partial c_n \, \partial \bar{c}_n} & \left(\overline{\frac{\partial f}{\partial c_n}}\right) \\ \frac{\partial f}{\partial c_1} & & \frac{\partial f}{\partial c_n} & f_c \end{bmatrix} \begin{bmatrix} x_1 - c_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n - c_n \\ 1 \end{bmatrix} = 0 \tag{2.1}$$

is called the tangent quadric of f at $(c_1, ..., c_n)$. We shall study only the cases that at least one of the first or second derivatives is not zero. It is clear that the tangent plane of (2.1) at $(c_1, ..., c_n)$ is the same as the tangent plane of $f = 0$ at this point.

Let the matrix of (2.1) be A, $\xi = (x_1 - c_1 ... x_n - c_n)$, and $\eta = (0...0\ 1)$. Then by section 8 of [1]

$$\xi A \eta^* = 0 \tag{2.2}$$

is the tangent plane of (2.1) at $(c_1, ..., c_n)$. Here η^* is the conjugate transpose of η.

We easily see that (2.2) can be written as

$$\sum_{i=1}^{n} \frac{\partial f}{\partial c_i} (x_i - c_i) = 0 . \tag{2.3}$$

3. Matrices related to f

Besides A there are other matrices of some interest. We denote the matrix of the quadratic form of (2.1) by Q. The projection on the normal and tangent plane are of some interest. We denote the projection on the normal by P, and clearly $I - P$ is the projection on the tangent plane where I is the identity matrix. It is easy to see that $P = (P_{ij})$, where