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VERTEX POINTS OF FUNCTIONS

by Ali R. AMIR-MOÉZ

For f a real function of n variables, usually the Hessian matrix is studied in connection with Gaussian and mean curvatures of $f(x_1, \dots, x_n)$. In this paper we study other properties of f in a neighborhood of a point. In particular we get a method for obtaining vertex points of the function f . We also generalize the idea to some complex cases.

1. DEFINITIONS AND NOTATIONS

Let f a function of complex variables x_1, \dots, x_n be of class C'' in x_1, \dots, x_n , and $\bar{x}_1, \dots, \bar{x}_n$, in a neighborhood of a point. Then f is called unitarily analytic if

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = \overline{\left(\frac{\partial^2 f}{\partial \bar{x}_i \partial x_j} \right)}.$$

Theorem: Let f be of class C'' in $x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n$ in a neighborhood of a point, and

$$\frac{\partial f}{\partial \bar{x}_k} = \overline{\left(\frac{\partial f}{\partial x_k} \right)}.$$

Then f is unitarily analytic.

The proof is quite simple and we omit it. Note that the converse is not necessarily true.

2. TANGENT QUADRIC

Let f be unitarily analytic in a neighborhood of (c_1, \dots, c_n) .

Let, for example, $\frac{\partial f}{\partial c_1}$ be the value of $\frac{\partial f}{\partial x_1}$ at (c_1, \dots, c_n) , and

$f_c = f(c_1, \dots, c_n)$. Then

$$(x_1 - c_1 \dots x_n - c_n) \begin{bmatrix} \frac{\partial^2 f}{\partial c_1 \partial \bar{c}_1} & \dots & \frac{\partial^2 f}{\partial c_1 \partial \bar{c}_n} \left(\frac{\partial f}{\partial c_1} \right) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial^2 f}{\partial c_n \partial \bar{c}_1} & & \frac{\partial^2 f}{\partial c_n \partial \bar{c}_n} \left(\frac{\partial f}{\partial c_n} \right) \\ \frac{\partial f}{\partial c_1} & & \frac{\partial f}{\partial c_n} & f_c \end{bmatrix} \begin{bmatrix} x_1 - c_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n - c_n \\ 1 \end{bmatrix} = 0 \quad (2.1)$$

is called the tangent quadric of f at (c_1, \dots, c_n) . We shall study only the cases that at least one of the first or second derivatives is not zero. It is clear that the tangent plane of (2.1) at (c_1, \dots, c_n) is the same as the tangent plane of $f = 0$ at this point.

Let the matrix of (2.1) be A , $\xi = (x_1 - c_1 \dots x_n - c_n)$, and $\eta = (0 \dots 0 \ 1)$. Then by section 8 of [1]

$$\xi A \eta^* = 0 \quad (2.2)$$

is the tangent plane of (2.1) at (c_1, \dots, c_n) . Here η^* is the conjugate transpose of η .

We easily see that (2.2) can be written as

$$\sum_{i=1}^n \frac{\partial f}{\partial c_i} (x_i - c_i) = 0. \quad (2.3)$$

3. MATRICES RELATED TO f

Besides A there are other matrices of some interest. We denote the matrix of the quadratic form of (2.1) by Q . The projection on the normal and tangent plane are of some interest. We denote the projection on the normal by P , and clearly $I - P$ is the projection on the tangent plane where I is the identity matrix. It is easy to see that $P = (P_{ij})$, where