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VERTEX POINTS OF FUNCTIONS

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For f a real function of n variables, usually the Hessian matrix is studied in connection with Gaussian and mean curvatures of $f(x_1, \ldots, x_n)$. In this paper we study other properties of f in a neighborhood of a point. In particular we get a method for obtaining vertex points of the function f. We also generalize the idea to some complex cases.

1. Definitions and notations

Let f a function of complex variables $x_1, \ldots, x_n$ be of class C'' in $x_1, \ldots, x_n$, and $\bar{x}_1, \ldots, \bar{x}_n$, in a neighborhood of a point. Then f is called unitarily analytic if

$$\frac{\partial^2 f}{\partial x_i \, \partial \bar{x}_j} = \left(\overline{\frac{\partial^2 f}{\partial \bar{x}_i \, \partial x_j}}\right).$$

Theorem: Let f be of class C'' in $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n$ in a neighborhood of a point, and

$$\frac{\partial f}{\partial \bar{x}_k} = \left(\overline{\frac{\partial f}{\partial x_k}}\right).$$

Then f is unitarily analytic.

The proof is quite simple and we omit it. Note that the converse is not necessarily true.

2. Tangent Quadric

Let f be unitarily analytic in a neighborhood of $(c_1, \ldots, c_n)$.

Let, for example, $\dfrac{\partial f}{\partial c_1}$ be the value of $\dfrac{\partial f}{\partial x_1}$ at $(c_1, \ldots, c_n)$, and

$f_c = f(c_1, \dots, c_n)$. Then

$$(x_1-c_1 \dots x_n-c_n)\begin{bmatrix} \dfrac{\partial^2 f}{\partial c_1\, \partial \bar{c}_1} & \cdots & \dfrac{\partial^2 f}{\partial c_1\, \partial \bar{c}_n} & \left(\overline{\dfrac{\partial f}{\partial c_1}}\right) \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \dfrac{\partial^2 f}{\partial c_n\, \partial \bar{c}_1} & & \dfrac{\partial^2 f}{\partial c_n\, \partial \bar{c}_n} & \left(\overline{\dfrac{\partial f}{\partial c_n}}\right) \\ \dfrac{\partial f}{\partial c_1} & & \dfrac{\partial f}{\partial c_n} & f_c \end{bmatrix}\begin{bmatrix} x_1-c_1 \\ \cdot \\ \cdot \\ \cdot\cdot \\ x_n-c_n \\ 1 \end{bmatrix} = 0 \tag{2.1}$$

is called the tangent quadric of f at $(c_1, \dots, c_n)$. We shall study only the cases that at least one of the first or second derivatives is not zero. It is clear that the tangent plane of (2.1) at $(c_1, \dots, c_n)$ is the same as the tangent plane of $f = 0$ at this point.

Let the matrix of (2.1) be A, $\xi = (x_1-c_1 \dots x_n-c_n)$, and $\eta = (0 \dots 0\ 1)$. Then by section 8 of [1]

$$\xi A \eta^* = 0 \tag{2.2}$$

is the tangent plane of (2.1) at $(c_1, \dots, c_n)$. Here η^* is the conjugate transpose of η.

We easily see that (2.2) can be written as

$$\sum_{i=1}^{n} \frac{\partial f}{\partial c_i}(x_i-c_i) = 0 . \tag{2.3}$$

3. Matrices related to f

Besides A there are other matrices of some interest. We denote the matrix of the quadratic form of (2.1) by Q. The projection on the normal and tangent plane are of some interest. We denote the projection on the normal by P, and clearly $I-P$ is the projection on the tangent plane where I is the identity matrix. It is easy to see that $P = (P_{ij})$, where

$$P_{ij} = \frac{\overline{\left(\frac{\partial f}{\partial x_i}\right)} \frac{\partial f}{\partial x_j}}{\sum \left| \frac{\partial f}{\partial x_i} \right|^2} \, .$$

This is proved by considering the inner product of a vector $\xi = (x_1, \ldots, x_n)$ and a unit vector on

$$\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) .$$

3. Quadric curvature

If (2.1) becomes of the form

$$[\sum a_i (x_i - c_i)] \, [\sum \bar{a}_i (\bar{x}_i - \bar{c}_i)] = 0 \, ,$$

then $f(x_1, \ldots, x_n)$ is called doubly flat at $(c_1, \ldots, c_n)$. Suppose (2.1) does not have this form. Then by sec 6 of [1] centers $(x_1, \ldots, x_n)$ of (2.1) may be obtained by

$$\xi Q = - \left(\frac{\partial f}{\partial \xi}\right) , \tag{3.1}$$

where the row matrix ξ is:

$$\xi = (x_1 - c_1 \ldots x_n - c_n) \, , \; and \; \frac{\partial f}{\partial \xi} = \left(\frac{\partial f}{\partial c_1} \cdots \frac{\partial f}{\partial c_n}\right) .$$

The equation (3.1) is a system of n linear equations in n unknowns.

The following cases may occur:

I. Let Q be non-singular. Then the quadric has a unique center which is called the center of quadric curvature of $f(x_1, \ldots, x_n)$ at $\gamma = (c_1, \ldots, c_n)$. Let

$$\xi = - \left(\frac{\partial f}{\partial \xi}\right) Q^{-1} \, .$$

Then the center is the point defined by $\xi - \gamma$.

II. Let the rank of Q be k, and centers exist. Then these centers are solutions of

$$\xi_k = \xi E = -\left(\frac{\partial f}{\partial \xi}\right) E Q^{-1}, \qquad (3.2)$$

where Q^{-1} is the reciprocal of Q, see [2]. That is, if E is the projection on the range of Q, then

$$Q^{-1} Q = Q Q^{-1} = E.$$

Here we choose the center of quadric curvatures at a point of (3.2) so that, it is at the shortest distance from γ.

III. When the rank of Q is k and the quadric does not have centers, then we say that f does not have a center of quadric curvature.

4. Direction of quadric curvature

In part I and II of section 3 we respectively call the vectors ξ and ξ_k the directions of quadric curvature of f at $(c_1, \ldots, c_n)$. In III of section 3, we define the direction of quadric curvature to be a vector δ which satisfies

$$\delta = \delta E = -\left(\frac{\partial f}{\partial \xi}\right) E Q^{-1},$$

where E is the projection described in section 3.

5. Vertex points

Let at the point $\gamma = (c_1, \ldots, c_n)$ of f the direction of quadric curvature be the same as the normal to $f = 0$. Then γ is called a vertex point of the function f.

Theorem: A necessary and sufficient condition for a point to be a vertex point of the function f is that at that point

$$PQ = QP,$$

where P and Q are the matrices described in section 3.

Proof: At a vertex point the projection of the direction of quadric curvature on the tangent plane is zero. Thus

$$- \left(\frac{\partial f}{\partial \xi}\right) Q^{-1} (I - P) = 0 .$$

This implies that

$$Q^{-1} P Q = P .$$

In all cases this implies

$$PQ = QP .$$

A vertex point in particular may become a spherical point, i.e. a point where

$$\frac{\partial^2 f}{\partial x_i \, \partial \bar{x}_j} = \lambda \delta_{ij} , \lambda$$

is a constant.

A vertex point will be called a cylindrical point when

$$\frac{\partial^2 f}{\partial x_i \, \partial \bar{x}_j} = \lambda \delta_{ij} , i , j \leqq k ,$$

$$\frac{\partial^2 f}{\partial x_i \, \partial \bar{x}_j} = 0 , i , j > k .$$

6. Functions of fixed center

An interesting fact about these functions is that they are not necessarily quadrics.

The equation.

$$\xi Q = - \left(\frac{\partial f}{\partial \xi}\right) \tag{6.1}$$

where $\xi = (c_1 - x_1, \ldots, c_n - x_n)$, and $(c_1, \ldots, c_n)$ is the fixed center gives f. To produce a counter example we let the origin be the center and the dimension of the space be two. Then in the real case (6.1) becomes

$$\left.\begin{aligned} x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial f}{\partial x} \\ x\frac{\partial^2 f}{\partial x \partial y} + y\frac{\partial^2 f}{\partial y^2} &= \frac{\partial f}{\partial y}\,. \end{aligned}\right\} \tag{6.2}$$

We can easily find a solution of (6.2) which is not a quadric. For example

$$f = \frac{x^2}{2}\log\left(\frac{\sqrt{x^2+y^2}+y}{x}\right) + \frac{y}{2}\sqrt{x^2+y^2}\,.$$

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