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# VERTEX POINTS OF FUNCTIONS

by Ali R. AMIR-MOÉZ

For  $f$  a real function of  $n$  variables, usually the Hessian matrix is studied in connection with Gaussian and mean curvatures of  $f(x_1, \dots, x_n)$ . In this paper we study other properties of  $f$  in a neighborhood of a point. In particular we get a method for obtaining vertex points of the function  $f$ . We also generalize the idea to some complex cases.

## 1. DEFINITIONS AND NOTATIONS

Let  $f$  a function of complex variables  $x_1, \dots, x_n$  be of class  $C''$  in  $x_1, \dots, x_n$ , and  $\bar{x}_1, \dots, \bar{x}_n$ , in a neighborhood of a point. Then  $f$  is called unitarily analytic if

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = \overline{\left( \frac{\partial^2 f}{\partial \bar{x}_i \partial x_j} \right)}.$$

*Theorem:* Let  $f$  be of class  $C''$  in  $x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n$  in a neighborhood of a point, and

$$\frac{\partial f}{\partial \bar{x}_k} = \overline{\left( \frac{\partial f}{\partial x_k} \right)}.$$

Then  $f$  is unitarily analytic.

The proof is quite simple and we omit it. Note that the converse is not necessarily true.

## 2. TANGENT QUADRIC

Let  $f$  be unitarily analytic in a neighborhood of  $(c_1, \dots, c_n)$ .

Let, for example,  $\frac{\partial f}{\partial c_1}$  be the value of  $\frac{\partial f}{\partial x_1}$  at  $(c_1, \dots, c_n)$ , and

$f_c = f(c_1, \dots, c_n)$ . Then

$$(x_1 - c_1 \dots x_n - c_n) \begin{bmatrix} \frac{\partial^2 f}{\partial c_1 \partial \bar{c}_1} & \dots & \frac{\partial^2 f}{\partial c_1 \partial \bar{c}_n} \left( \frac{\partial f}{\partial c_1} \right) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial^2 f}{\partial c_n \partial \bar{c}_1} & & \frac{\partial^2 f}{\partial c_n \partial \bar{c}_n} \left( \frac{\partial f}{\partial c_n} \right) \\ \frac{\partial f}{\partial c_1} & & \frac{\partial f}{\partial c_n} & f_c \end{bmatrix} \begin{bmatrix} x_1 - c_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n - c_n \\ 1 \end{bmatrix} = 0 \quad (2.1)$$

is called the tangent quadric of  $f$  at  $(c_1, \dots, c_n)$ . We shall study only the cases that at least one of the first or second derivatives is not zero. It is clear that the tangent plane of (2.1) at  $(c_1, \dots, c_n)$  is the same as the tangent plane of  $f = 0$  at this point.

Let the matrix of (2.1) be  $A$ ,  $\xi = (x_1 - c_1 \dots x_n - c_n)$ , and  $\eta = (0 \dots 0 \ 1)$ . Then by section 8 of [1]

$$\xi A \eta^* = 0 \quad (2.2)$$

is the tangent plane of (2.1) at  $(c_1, \dots, c_n)$ . Here  $\eta^*$  is the conjugate transpose of  $\eta$ .

We easily see that (2.2) can be written as

$$\sum_{i=1}^n \frac{\partial f}{\partial c_i} (x_i - c_i) = 0. \quad (2.3)$$

### 3. MATRICES RELATED TO $f$

Besides  $A$  there are other matrices of some interest. We denote the matrix of the quadratic form of (2.1) by  $Q$ . The projection on the normal and tangent plane are of some interest. We denote the projection on the normal by  $P$ , and clearly  $I - P$  is the projection on the tangent plane where  $I$  is the identity matrix. It is easy to see that  $P = (P_{ij})$ , where

$$P_{ij} = \frac{\left(\frac{\partial f}{\partial x_i}\right) \frac{\partial f}{\partial x_j}}{\sum \left|\frac{\partial f}{\partial x_i}\right|^2}.$$

This is proved by considering the inner product of a vector  $\xi = (x_1, \dots, x_n)$  and a unit vector on

$$\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

### 3. QUADRIC CURVATURE

If (2.1) becomes of the form

$$[\sum a_i (x_i - c_i)] [\sum \bar{a}_i (\bar{x}_i - \bar{c}_i)] = 0,$$

then  $f(x_1, \dots, x_n)$  is called doubly flat at  $(c_1, \dots, c_n)$ . Suppose (2.1) does not have this form. Then by sec 6 of [1] centers  $(x_1, \dots, x_n)$  of (2.1) may be obtained by

$$\xi Q = - \left(\frac{\partial f}{\partial \xi}\right), \tag{3.1}$$

where the row matrix  $\xi$  is:

$$\xi = (x_1 - c_1 \dots x_n - c_n), \text{ and } \frac{\partial f}{\partial \xi} = \left(\frac{\partial f}{\partial c_1} \dots \frac{\partial f}{\partial c_n}\right).$$

The equation (3.1) is a system of  $n$  linear equations in  $n$  unknowns.

The following cases may occur:

I. Let  $Q$  be non-singular. Then the quadric has a unique center which is called the center of quadric curvature of  $f(x_1, \dots, x_n)$  at  $\gamma = (c_1, \dots, c_n)$ . Let

$$\xi = - \left(\frac{\partial f}{\partial \xi}\right) Q^{-1}.$$

Then the center is the point defined by  $\xi - \gamma$ .

II. Let the rank of  $Q$  be  $k$ , and centers exist. Then these centers are solutions of

$$\xi_k = \xi E = - \left( \frac{\partial f}{\partial \xi} \right) EQ^{-1}, \quad (3.2)$$

where  $Q^{-1}$  is the reciprocal of  $Q$ , see [2]. That is, if  $E$  is the projection on the range of  $Q$ , then

$$Q^{-1} Q = QQ^{-1} = E.$$

Here we choose the center of quadric curvatures at a point of (3.2) so that, it is at the shortest distance from  $\gamma$ .

III. When the rank of  $Q$  is  $k$  and the quadric does not have centers, then we say that  $f$  does not have a center of quadric curvature.

#### 4. DIRECTION OF QUADRIC CURVATURE

In part I and II of section 3 we respectively call the vectors  $\xi$  and  $\xi_k$  the directions of quadric curvature of  $f$  at  $(c_1, \dots, c_n)$ . In III of section 3, we define the direction of quadric curvature to be a vector  $\delta$  which satisfies

$$\delta = \delta E = - \left( \frac{\partial f}{\partial \xi} \right) EQ^{-1},$$

where  $E$  is the projection described in section 3.

#### 5. VERTEX POINTS

Let at the point  $\gamma = (c_1, \dots, c_n)$  of  $f$  the direction of quadric curvature be the same as the normal to  $f = 0$ . Then  $\gamma$  is called a vertex point of the function  $f$ .

*Theorem:* A necessary and sufficient condition for a point to be a vertex point of the function  $f$  is that at that point

$$PQ = QP,$$

where  $P$  and  $Q$  are the matrices described in section 3.

Proof: At a vertex point the projection of the direction of quadric curvature on the tangent plane is zero. Thus

$$-\left(\frac{\partial f}{\partial \xi}\right) Q^{-1} (I - P) = 0.$$

This implies that

$$Q^{-1} P Q = P.$$

In all cases this implies

$$P Q = Q P.$$

A vertex point in particular may become a spherical point, i.e. a point where

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = \lambda \delta_{ij}, \lambda$$

is a constant.

A vertex point will be called a cylindrical point when

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = \lambda \delta_{ij}, i, j \leq k,$$

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = 0, i, j > k.$$

## 6. FUNCTIONS OF FIXED CENTER

An interesting fact about these functions is that they are not necessarily quadrics.

The equation.

$$\xi Q = -\left(\frac{\partial f}{\partial \xi}\right) \tag{6.1}$$

where  $\xi = (c_1 - x_1, \dots, c_n - x_n)$ , and  $(c_1, \dots, c_n)$  is the fixed center gives  $f$ . To produce a counter example we let the origin be the center and the dimension of the space be two. Then in the real case (6.1) becomes

$$\left. \begin{aligned} x \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial f}{\partial x} \\ x \frac{\partial^2 f}{\partial x \partial y} + y \frac{\partial^2 f}{\partial y^2} &= \frac{\partial f}{\partial y} \end{aligned} \right\} \quad (6.2)$$

We can easily find a solution of (6.2) which is not a quadric. For example

$$f = \frac{x^2}{2} \log \left( \frac{\sqrt{x^2 + y^2} + y}{x} \right) + \frac{y}{2} \sqrt{x^2 + y^2}.$$

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