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# ARITHMETICAL NOTES, XI. SOME DIVISOR IDENTITIES

by Eckford COHEN

1. *Introduction.* In a series of papers [1, 2, 3, 4], the author has discussed arithmetical functions related to the unitary divisors  $d$  of a positive integer  $r$ , that is, divisors  $d$  of  $r$ , such that  $(d, \delta) = 1$ , where  $\delta$  is the complementary divisor of  $r$ . It is the purpose of this note to derive in a unified manner some of the basic identities proved in these papers.

The second section is concerned with the unitary analogue  $c^*(n, r)$  of Ramanujan's trigonometric sum  $c(n, r)$ , § 3 with the unitary analogue of Möbius inversion, and § 4 with orthogonality properties of  $c^*(n, r)$ .

2. *The sum  $c^*(n, r)$ .* We recall that for integers  $n$ ,  $c(n, r)$  is defined by

$$c(n, r) = \sum_{(x, r)=1} e(nx, r), \quad e(n, r) = \exp(2\pi in/r), \quad (1)$$

where the summation is over a reduced residue system (mod  $r$ ). Let us define  $(n, r)_*$  to be the largest unitary divisor of  $r$  which is a factor of  $n$ . Analogous to (1) we place [1, § 2]

$$c^*(n, r) = \sum_{(x, r)_*=1} e(nx, r), \quad (2)$$

where the summation is over those integers  $x$  (mod  $r$ ) such that  $(x, r)_* = 1$ . Such a system of numbers is said to form a semi-reduced residue system (mod  $r$ ).

We first express  $c^*(n, r)$  in terms of  $c(n, r)$ . Let  $\gamma(r)$  denote the largest divisor of  $r$  with no square factors other than 1.

*Identity 1* ([3, (3.1)]).

$$c^*(n, r) = \sum_{\substack{d|r \\ \gamma(d)=\gamma(r)}} c(n, d). \quad (3)$$

*Proof.* Classifying the  $x$  in (2) according to their greatest common divisor with  $r$ , one obtains

$$\begin{aligned} c^*(n, r) &= \sum_{d\delta=r} \sum_{\substack{(x, r)=\delta \\ x \pmod{r}}} e(nx, r) = \sum_{d\delta=r} \sum_{(X, d)=1} e(n\delta X, r) \\ &\quad \delta \left| \frac{r}{\gamma(r)} \right. \quad (x=\delta X) \qquad \delta \left| \frac{r}{\gamma(r)} \right. \\ &= \sum_{\substack{d\delta=r \\ \delta \left| \frac{r}{\gamma(r)} \right.}} \sum_{(X, d)=1} e(nX, d) = \sum_{\substack{d|r \\ \gamma(r)|d}} c(n, d), \end{aligned}$$

which is the same as (3).

Let  $\phi(r)$  denote the Euler  $\phi$ -function and  $\mu(r)$  the Möbius function. It is well known that

$$c(n, r) = \sum_{d|(n, r)} d\mu\left(\frac{r}{d}\right). \quad (4)$$

We also note that

$$\phi(r) = c(0, r) = \sum_{d\delta=r} d\mu(\delta), \quad \mu(r) = c(1, r). \quad (5)$$

*Definition.* Place

$$\phi^*(r) = c^*(0, r), \quad \mu^*(r) = c^*(1, r), \quad (6)$$

so that  $\phi^*(r)$  is the number of integers in a semi-reduced residue system  $(\text{mod } r)$ .

As corollaries of (3) we obtain the following two formulas, by virtue of (5).

*Identity 2* (cf. [2, Lemma 3.1,  $k = 1$ ]).

$$\phi^*(r) = \sum_{\substack{d|r \\ \gamma(d)=\gamma(r)}} \phi(d). \quad (7)$$

*Identity 3* ([1, (2.9)], [3, (3.5)]).

$$\mu^*(r) = \sum_{\substack{d|r \\ \gamma(d)=\gamma(r)}} \mu(d) = \mu(\gamma(r)). \quad (8)$$

*Notation.* Let  $d \parallel r$  and  $d^* \delta = r$  be used to signify that  $d$  is a unitary divisor of  $r$ .

We shall need the following relation for a proof of the unitary analogue of (4).

LEMMA 1. Let  $k$  be a divisor of  $r$ . Then

$$\sum_{\substack{d \mid \frac{r}{k} \\ \gamma(r) = \gamma(dk)}} \mu(d) = \begin{cases} \mu^*(r/k) & \text{if } k \parallel r, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Remark 1. We first recall that

$$\sum_{d \mid r} \mu(d) = \varepsilon(r) = \begin{cases} 1 & (r = 1) \\ 0 & (r > 1), \end{cases} \quad (10)$$

and that  $\mu(r) = 0$  unless  $r$  is square-free.

*Proof.* Let  $\sum$  denote the sum in (9).

Case 1 ( $k \parallel r$ ). In this case, by the above remark,

$$\sum = \sum_{\substack{d \mid \frac{r}{k} \\ \gamma(r) = \gamma(k)\gamma(d)}} \mu(d) = \sum_{\substack{d \mid \frac{\gamma(r)}{\gamma(k)} \\ d\gamma(k) = \gamma(r)}} \mu(d) = \mu\left(\frac{\gamma(r)}{\gamma(k)}\right) = \mu\left(\gamma\left(\frac{r}{k}\right)\right),$$

and the first part of (9) results by (8).

Case 2 ( $k \nparallel r$ ). In this case let  $r_1$  be the largest common unitary divisor of  $k$  and  $r$ ,  $r = r_1^* r_2$ ,  $k = r_1^* k_2$ , so that  $k_2 \mid r_2$ ,  $r_2/k_2 > 1$ , and  $\gamma(r_2) = \gamma(k_2) = \gamma(r_2/k_2)$ . Hence by Remark 1,

$$\sum = \sum_{\substack{d \mid \frac{r_2}{k_2} \\ \gamma(r_2) = \gamma(k_2 d)}} \mu(d) = \sum_{d \mid \frac{r_2}{k_2}} \mu(d) = 0.$$

Identity 4 ([1, (2.7)]).

$$c^*(n, r) = \sum_{d \parallel (n, r)^*} d \mu^*\left(\frac{r}{d}\right). \quad (11)$$

*Proof.* By Identity 1, (4), and Lemma 1, one obtains

$$\begin{aligned} c^*(n, r) &= \sum_{\substack{d\delta = r \\ \gamma(d) = \gamma(r)}} \sum_{D \mid (n, d)} D \mu\left(\frac{d}{D}\right) = \sum_{D \mid (n, r)} D \sum_{\substack{DE = d \\ d\delta = r \\ \gamma(r) = \gamma(d)}} \mu(E) \\ &= \sum_{D \mid (n, r)} D \sum_{\substack{E\delta = r/D \\ \gamma(r) = \gamma(DE)}} \mu(E) = \sum_{\substack{D \mid (n, r) \\ D \parallel r}} D \mu^*\left(\frac{r}{D}\right), \end{aligned}$$

which is the same as (11).

In the special case  $n = 0$ , one gets

*Identity 5* ([1, (2.8)]).

$$\vartheta^*(r) = \sum_{d \parallel r} d \mu^*\left(\frac{r}{d}\right). \quad (12)$$

3. *Unitary inversion.* In this section the following analogue of (10) is basic.

*Identity 6* ([1, (2.5)]).

$$\sum_{d \parallel r} \mu^*(d) = \epsilon(r). \quad (13)$$

*Remark 2.* As  $d$  ranges over the unitary divisors of  $r$ ,  $\gamma(d)$  ranges over all divisors of  $\gamma(r)$ .

*Proof.* By Remark 2, (8), and (10), it follows that

$$\sum_{d \parallel r} \mu^*(d) = \sum_{d \parallel r} \mu(\gamma(d)) = \sum_{d | \gamma(r)} \mu(d) = \epsilon(\gamma(r)),$$

which proves (13).

We define the unitary product  $f^*g$  of two arithmetical functions  $f, g$ , with values in the complex field, by

$$f^*g = \sum_{d^*\delta=r} f(d)g(\delta). \quad (14)$$

LEMMA 2. *The set of all arithmetical functions forms a semi-group S relative to the unitary product. The function  $\epsilon$  defined by (10) is the identity of S.*

*Proof.* Evidently  $\epsilon^*f = f^*\epsilon = f$  for every function  $f$ . Moreover, the associative law,  $f_1^*(f_2^*f_3) = (f_1^*f_2)^*f_3$ , is easily verified for arbitrary functions,  $f_1, f_2, f_3$ .

*Identity 7* [Inversion formula, [1, Theorem 2.3]]. *For functions  $f, g$  of S,*

$$f(r) = \sum_{d \parallel r} g(d) \Leftrightarrow g(r) = \sum_{d^*\delta=r} f(d) \mu^*(\delta). \quad (15)$$

*Proof.* Let  $I$  denote the function of  $S$  defined to have the value 1 for all  $r$ ,  $I(r) \equiv 1$ . Then (13) may be written  $\mu^* * I = \epsilon$ . Thus by Lemma 2,  $\mu^*$  is invertible in  $S$  with the (unique) inverse

$(\mu^*)^{-1} = I$ . Hence,  $g * I = f \xrightarrow{\leftarrow} g = f * \mu^*$ , which is merely a reformulation of (15).

We note that (11) can be rewritten as

$$c^*(n, r) = \sum_{d^* \delta = r} \epsilon_d(n) \mu^*(\delta), \quad \epsilon_r(n) = \begin{cases} 1 & (r | n) \\ 0 & (r \nmid n). \end{cases} \quad (16)$$

Application of the inversion formula to (16) leads to

*Identity 8* ([1, (2.2)]).

$$\sum_{d \parallel r} c^*(n, d) = \epsilon_r(n). \quad (17)$$

Noting that  $\epsilon_r(1) = \epsilon(r)$ , the relation (17) reduces to (13) in case  $n = 1$ . In the case  $n = 0$ , we have

*Identity 9* ([1, (2.4)]).

$$\sum_{d \parallel r} \vartheta^*(d) = r. \quad (18)$$

The latter result can be deduced independently on applying the unitary inversion formula to (12).

4. *Orthogonality properties.* In [3, Theorem 3.2] it was proved that for unitary divisors  $d_1, d_2$  of  $r$ ,

$$\sum_{n \equiv a+b \pmod{r}} c^*(a, d_1) c^*(b, d_2) = \begin{cases} rc^*(n, e) & \text{if } e = d_1 = d_2, \\ 0 & \text{if } d_1 \neq d_2, \end{cases} \quad (19)$$

the summation on the left extending over all  $a, b \pmod{r}$  such that  $n \equiv a + b$ . This result arose from an analogous relation satisfied by  $c(n, r)$  [3, (1.4)] on application of (3). We deduce now a number of consequences of (19).

Letting  $n = 0$  and noting that  $c^*(-n, r) = c^*(n, r)$ , (19) becomes.

*Identity 10.* If  $d_1 \parallel r, d_2 \parallel r$ , then

$$\sum_{a \pmod{r}} c^*(a, d_1) c^*(a, d_2) = \begin{cases} r\vartheta^*(e) & \text{if } e = d_1 = d_2, \\ 0 & \text{if } d_1 \neq d_2. \end{cases} \quad (20)$$

We prove next

*Identity 11.* If  $d_1 \parallel r, d_2 \parallel r$ , then

$$\sum_{d*\delta=r} c^*(d, d_1) c^*(d, d_2) \vartheta^*(\delta) = \begin{cases} r\vartheta^*(e) & \text{if } e = d_1 = d_2, \\ 0 & \text{if } d_1 \neq d_2. \end{cases} \quad (21)$$

*Remark 3.* If  $(e, r)_* = 1$ , then  $c^*(ne, r) = c^*(n, r)$ .

*Remark 4.* ([4, Remark 2.1]). If  $d \parallel r$ , then any semi-reduced residue system (mod  $r$ ) contains such a system (mod  $d$ ).

*Proof.* Let the left member of (20) be denoted  $\Sigma$ . With  $(a, r)_* = d$ , we may write  $a = dX, (X, r/d)_* = 1$ . By Remark 4, one may suppose  $X \pmod{r/d}$  chosen so that  $(X, r)_* = 1$ . Hence, since  $d_1$  and  $d_2$  are unitary divisors of  $r$ , it follows by Remark 3 that

$$\Sigma = \sum_{\substack{d \parallel r \\ (X, r)_* = 1 \\ X \pmod{r/d}}} c^*(dX, d_1) c^*(dX, d_2) = \sum_{d \parallel r} c^*(d, d_1) c^*(d, d_2) \sum_{(X, r/d)_* = 1} 1,$$

which is the left of (21). The proof is complete, by Identity 10.

*Remark 5* ([1, Corollary 2.2.1, also cf. Lemma 6.1]). The function  $\phi^*(r)$  is multiplicative.

We require a simple formula proved in [4, (2.2)], namely,

$$\vartheta^*(e_1) c^*\left(\frac{r}{e_1}, e_2\right) = \vartheta^*(e_2) c^*\left(\frac{r}{e_2}, e_1\right), \quad e_1 \parallel r, e_2 \parallel r. \quad (22)$$

Applying (22) to the second factor in the sum in (21), it results that

*Identity 12* ([4, Lemma 2.4]). If  $d_1 \parallel r, d_2 \parallel r$ , then

$$\sum_{\delta \parallel r} c^*\left(\frac{r}{\delta}, d_1\right) c^*\left(\frac{r}{d_2}, \delta\right) = \begin{cases} r & \text{if } d_1 = d_2, \\ 0 & \text{if } d_1 \neq d_2. \end{cases} \quad (23)$$

*Identity 13* ([3, Theorem 3.3]). If  $m$  and  $n$  are integers, then

$$\sum_{\delta \parallel r} \frac{c^*(m, \delta) c^*(n, \delta)}{\vartheta^*(\delta)} = \begin{cases} \left(\frac{r}{\vartheta^*(r)}\right) \vartheta^*((n, r)_*) & \text{if } (m, r)_* = (n, r)_*, \\ 0 & \text{if } (m, r)_* \neq (n, r)_*. \end{cases} \quad (24)$$

*Proof.* Apply (22) to the first factor in the sum in (23), with  $d_1 = r/(m, r)_*$ ,  $d_2 = r/(n, r)_*$ , and use Remarks 3 and 5.

The following relation results from (24) in the case  $m = n$ ,

$$\sum_{d \parallel r} \frac{(c^*(n, d))^2}{\varnothing^*(d)} = \left( \frac{r}{\varnothing^*(r)} \right) \varnothing^*((n, r)_*). \quad (25)$$

Further, the Inversion Theorem and Remark 5 give

$$(c^*(n, r))^2 = \sum_{d \cdot \delta = r} d \varnothing^*((n, d)_*) \varnothing^*(\delta) \mu^*(\delta). \quad (26)$$

### BIBLIOGRAPHY

1. Eckford COHEN, Arithmetical functions associated with the unitary divisors of an integer, *Mathematische Zeitschrift*, Vol. 74 (1960), pp. 66-80.
2. ——— Eckford COHEN, An elementary method in the asymptotic theory of numbers, *Duke Mathematical Journal*, Vol. 28 (1961), pp. 183-192.
3. ——— Unitary functions (mod  $r$ ), *Duke Mathematical Journal*, Vol. 28 (1961), pp. 475-486.
4. ——— *Unitary functions (mod  $r$ )*, II, to appear.

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