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ON THE ASYMPTOTIC BEHAVIOR OF THE SUM OF A " NON-HARMONIC FOURIER SERIES "[1])

by P. M. ANSELONE

1. *Introduction.* This paper is concerned primarily with the asymptotic behavior as $t \to \infty$ of a function of the form

$$\varphi(t) = \sum_{n=-\infty}^{\infty} c_n e^{z_n t}, \quad t \geqq 0, \begin{cases} c_n = a_n + ib_n, \\ z_n = x_n + iy_n, \end{cases} \tag{1}$$

where

$$x_n < 0, \qquad \forall n, \tag{2}$$

$$M = \sup_n | z_n - 2\pi ni | < ln 2. \tag{3}$$

Conditions (2) and (3) will be generalized later. Note that if $z_n = 2\pi ni$, $\forall n$, then (1) is a Fourier series. In particular, a series of the type (1) may occur as the residue evaluation (Heaviside expansion) of the inverse Laplace transform integral

$$\varphi(t) = \lim_{\eta \to \infty} \frac{1}{2\pi i} \int_{\xi - i\eta}^{\xi + i\eta} \hat{\varphi}(z) e^{zt} dz, \quad t \geqq 0, \quad \xi > 0, \tag{4}$$

if the transform $\hat{\varphi}(z)$ is regular for $Re(z) \geqq 0$. In fact, such an example, which will be discussed later, motivated the present study.

Since $| e^{z_n t} | = e^{x_n t}$ and $x_n < 0$, it may seem plausible that $\varphi(t) \to 0$ as $t \to \infty$ or, at least, that $\varphi(t)$ is bounded. However, Rudin [6], constructed an example in which the series in (1) converges uniformly on each finite interval and $\varphi(t)$ is unbounded. In this paper we give conditions under which $\varphi(t) \to 0$ in certain mean-square senses as $t \to \infty$. Since the proof involves Hilbert

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space concepts, a resumé of some of the relevant theory is presented next.

2. *Biorthonormal Sequences.* Let H be a complex separable Hilbert space with the norm $\| \quad \|$ and the inner product $(\quad , \quad)$ [1]. A subset E of H is *complete* if the set of finite linear combinations of elements in E is dense in H. An equivalent condition is that

$$(f, g) = 0 \qquad \forall g \in E \Rightarrow f = 0 .$$

Let $\{h_n\}$ be an *orthonormal* sequence in H, i.e., $(h_m, h_n) = \delta_{mn}$. Recall that a series $\sum_n \gamma_n h_n$ converges to some $h \in H$ iff $\sum_n |\gamma_n|^2 < \infty$, in which case $\gamma_n = (h, h_n)$ and

$$h = \sum_n (h, h_n) h_n , \tag{5}$$

$$\| h \|^2 = \sum_n |(h, h_n)|^2 . \tag{6}$$

If $\{h_n\}$ is complete, then (5) and (6) hold for all $h \in H$ and (6) is *Parseval's identity.*

Let $\{f_n\}$ and $\{g_n\}$ be *biorthonormal* sequences in H, i.e., $(f_m, g_n) = \delta_{mn}$. If $h = \sum_n \gamma_n f_n$, then $\gamma_n = (h, g_n)$ and, hence,

$$h = \sum_n (h, g_n) f_n . \tag{7}$$

If $\{g_n\}$ is complete and the series $\sum_n (h, g_n) f_n$ converges for a particular $h \in H$, then (7) holds since

$$(h - \sum_n (h, g_n) f_n , g_m) = 0 , \qquad \forall m .$$

The same statements are valid with $\{f_n\}$ and $\{g_n\}$ interchanged.

Lemma 1. Let $\{f_n\}$ be a sequence and $\{h_n\}$ a complete orthonormal sequence in H. Suppose there exists a constant θ, $0 \leqq \theta < 1$, such that

$$\| \sum_{n \in F} \gamma_n (h_n - f_n) \|^2 \leqq \theta^2 \sum_{n \in F} |\gamma_n|^2 , \tag{8}$$

1) Cf. TAYLOR [7], pp. 118-119.

for each finite set F and arbitrary constants γ_n, $n \in F$. Then: $\{f_n\}$ is complete; there exists a unique complete sequence $\{g_n\}$ in H such that $\{f_n\}$ and $\{g_n\}$ are biorthonormal;

$$h = \sum_n (h, g_n) f_n = \sum_n (h, f_n) g_n \qquad \forall h \in H; \tag{9}$$

$$(1+\theta)^{-2} \| h \|^2 \leqq \sum_n |(h, g_n)|^2 \leqq (1-\theta)^{-2} \| h \|^2 \qquad \forall h \in H; \tag{10}$$

$$(1-\theta)^2 \| h \|^2 \leqq \sum_n |(h, f_n)|^2 \leqq (1+\theta)^2 \| h \|^2 \qquad \forall h \in H. \tag{11}$$

This fundamental result was proved for real $L_2(\alpha, \beta)$ by Paley and Wiener [4; pg. 100]. It was extended to a complex separable Hilbert space by Duffin and Eachus [3]. An independent proof was given by Nagy [5; pp. 208-210].

Lemma 2. Assume the hypotheses of Lemma 1. Then a series $\sum_n \gamma_n f_n$ converges to some $h \in H$ iff $\sum_n |\gamma_n|^2 < \infty$, in which case $\gamma_n = (h, g_n)$ and

$$(1+\theta)^{-2} \| h \|^2 \leqq \sum_n |\gamma_n|^2 \leqq (1-\theta)^{-2} \| h \|^2. \tag{12}$$

Proof. If $h = \sum_n \gamma_n h_n$, then $\gamma_n = (h, g_n)$, so that (10) implies (12) and $\sum_n |\gamma_n|^2 < \infty$. If $\sum_n |\gamma_n|^2 < \infty$, then, also by (10),

$$(1+\theta)^{-2} \| \sum_{n \in F} \gamma_n f_n \|^2 \leqq \sum_{n \in F} |\gamma_n|^2, \qquad F \text{ finite},$$

so that $\sum_n \gamma_n f_n$ converges by the Cauchy criterion.

Lemma 3. If (3) holds, then the hypotheses in Lemmas 1 and 2 are satisfied with $H = L_2(0, 1)$,

$$h_n(t) = e^{int}, \qquad f_n(t) = e^{z_n t}, \quad \begin{cases} 0 \leqq t \leqq 1, \\ n = 0, \pm 1, \ldots, \end{cases} \tag{13}$$

and $\theta = e^M - 1$.

Proof. Paley and Wiener [4; pp. 108-109] proved the analogous result for $L_2(-\pi, \pi)$ with $M < \pi^{-2}$ and $\theta = M\pi^2$. Duffin and Eachus [3] extended that result by taking $M < \frac{ln 2}{\pi}$ and $\theta = e^{M\pi} - 1$. Their proof goes over without essential change to the present situation.

3. *The Asymptotic Behavior of $\varphi(t)$.*

Theorem 1. Assume (2) and (3). Then the series in (1) converges in $L_2(0, s)$ for each $s > 0$ iff

$$\sum_n |c_n|^2 < \infty, \tag{14}$$

in which case

$$\int_s^{s+\tau} |\varphi(t)|^2 \, dt \to 0, \qquad as \ s \to \infty, \tag{15}$$

for each fixed $\tau > 0$, and

$$\frac{1}{s}\int_0^s |\varphi(t)|^2 \, dt \to 0, \qquad as \ s \to \infty. \tag{16}$$

Proof. According to Lemma 3, Lemmas 1 and 2 are applicable. Consider (1) in the form

$$\varphi(t+k) = \varphi_k(t) = \sum_n c_n e^{z_n k} f_n(t), \quad \begin{cases} 0 \leqq t \leqq 1, \\ k = 0, 1, \dots . \end{cases} \tag{17}$$

By Lemma 2, this series converges in $L_2(0, 1)$ for a particular k iff

$$\sum_n |c_n e^{z_n k}|^2 = \sum_n |c_n|^2 e^{2x_n k} < \infty, \tag{18}$$

in which case

$$(1+\theta)^{-2} \|\varphi_k\|^2 \leqq \sum_n |c_n|^2 e^{2x_n k} \leqq (1-\theta)^{-2} \|\varphi_k\|^2. \tag{19}$$

Since $x_n < 0$, (18) for $k = 0$ implies (18) for all k. Therefore, thə series in (17) converges in $L_2(0, 2\pi)$ for every k iff (14) is satisfied. This justifies the first assertion of the theorem.

Now assume (14). Since $x_n < 0$, the series in (18) converges uniformly with respect to k. It follows that

$$\sum_n |c_n|^2 e^{2x_n k} \to 0 \qquad as\ k \to \infty . \tag{20}$$

By (19) and (20),

$$\| \varphi_k \| \to 0 , \qquad as\ k \to \infty , \tag{21}$$

which implies (15) and (16).

To facilitate the next theorem, let $|||\ \ |||$ denote the norm for $L_2(0, \infty)$.

Theorem 2. Assume (2) and (3). Then these three conditions are equivalent: (A) the series in (1) converges in $L_2(0, s)$ for each $s > 0$, and $\varphi \in L_2(0, \infty)$; ($B$) the series in (1) converges in $L_2(0, \infty)$;

$$\sum_n \frac{|c_n|^2}{-x_n} < \infty . \tag{22}$$

If (A), (B) or (22) is satisfied, then

$$\frac{3}{4 ln 2} (1+\theta)^{-2} ||| \varphi |||^2 \leqq \sum_n \frac{|c_n|^2}{-x_n} \leqq 2(1-\theta)^{-2} ||| \varphi |||^2 . \tag{23}$$

Proof. By (2) and (3),

$$-ln 2 < x_n < 0 , \qquad \forall n . \tag{24}$$

Hence, (22) implies (14). In view of Theorem 1, we can assume without loss of generality in this proof that (14) is satisfied and, hence, that the series in (1) converges to $\varphi(t)$ in $L_2(0, s)$ for each $s > 0$.

Let

$$||| \varphi |||^2 = \int_0^\infty | \varphi(t) |^2 \, dt , \tag{25}$$

whether finite or infinite. By (17), (19) and (25),

$$||| \varphi |||^2 = \sum_{k=0}^{\infty} || \varphi_k ||^2 , \tag{26}$$

$$(1+\theta)^{-2} \, ||| \varphi |||^2 \leqq \sum_n \frac{|c_n|^2}{1-e^{2x_n}} \leqq (1-\theta)^{-2} \, ||| \varphi |||^2 . \quad (27)$$

For $s \neq 0$, $\dfrac{1-e^s}{-s}$ is positive, monotone increasing, and tends to 1 as $s \to 0$. Hence, by (24),

$$\frac{3}{4 ln 2} < \frac{1-e^{2x_n}}{-x_n} < 2 , \qquad \forall n . \quad (28)$$

This inequality and (27) imply (23). Therefore, $\varphi \in L_2(0, \infty)$ iff (22) holds. This proves the equivalence of (A) and (22).

For each finite set F of integers, let

$$\varphi^F(t) = \sum_{n \in F} c_n e^{z_n t} , \; t \geqq 0 . \quad (29)$$

Since φ^F is a special case of φ, (23) yields

$$\frac{3}{4 ln 2} (1+\theta)^{-2} \, ||| \varphi^F |||^2 \leqq \sum_{n \in F} \frac{|c_n|^2}{-x_n} \leqq 2(1-\theta)^{-2} \, ||| \varphi^F |||^2 .$$

It follows by means of the Cauchy criterion that (B) and (22) are equivalent. Thus, the theorem is proved.

As indicated in the foregoing proof, (22) implies (14). However, (14) does not imply (22). To see this, let $x_n \to 0$ as $n \to \infty$. Thus, the series in (1) may converge in $L_2(0, s)$ for each $s > 0$, but $\varphi \notin L_2(0, \infty)$.

The following elementary theorem is included for comparison with Theorem 2.

Theorem 3. If (2) holds and

$$\sum_n \frac{|c_n|}{\sqrt{-x_n}} < \infty , \quad (31)$$

then the series in (1) converges in $L_2(0, \infty)$ and

$$||| \varphi ||| \leqq \frac{1}{\sqrt{2}} \sum_n \frac{|c_n|}{\sqrt{-x_n}} . \quad (32)$$

Proof. Assume (31). Then, by (1), (25) and the Cauchy-Schwarz inequality,

$$||| \varphi |||^2 \leqq \sum_m \sum_n | c_m c_n | \int_0^\infty e^{x_m t} e^{x_n t} dt ,$$

$$||| \varphi |||^2 \leqq \sum_m \sum_n | c_m c_n | \left(\int_0^\infty e^{2x_m t} dt \right)^{\frac{1}{2}} \left(\int_0^\infty e^{2x_n t} dt \right)^{\frac{1}{2}} ,$$

$$||| \varphi |||^2 \leqq \sum_m \sum_n | c_m c_n | \left(\frac{1}{-2x_m} \right)^{\frac{1}{2}} \left(\frac{1}{-2x_n} \right)^{\frac{1}{2}} = \left[\sum_n \frac{| c_n |}{(-2x_n)^{\frac{1}{2}}} \right]^2 ,$$

which implies the desired results.

Since (31) implies (22), the first statement of Theorem 3 could have been obtained also from Theorem 2. Since (22) does not imply (31), Theorem 2 contains a stronger result.

4. *Generalizations*. Results similar to the above can be obtained under more general conditions. In place of (2) and (3), consider

$$x_n < \xi \qquad \forall n , \tag{33}$$

$$\sup_n | z_n - \xi - \lambda n i | < \frac{\lambda \ln 2}{2\pi} , \tag{34}$$

for some real ξ and some $\lambda > 0$. The changes of variable

$$t^* = \frac{\lambda t}{2\pi} , \qquad \varphi^*(t^*) = e^{-\xi t} \varphi(t) , \qquad z_n^* = \frac{2\pi}{\lambda}(z_n - \xi) , \tag{35}$$

yield (1), (2) and (3) in terms of $\varphi^*(t^*)$ and $z_n^* = x_n^* + i y_n^*$. Thus, the theorems of Section 3 apply. By means of (35), the statements of these theorems can be expressed in terms of $\varphi(t)$ and z_n. Since only substitutions are involved, we omit further details.

Consider (3) or (34) with $\sup_n$ replaced by $\limsup_{n\to\infty}$. This case can be treated by expressing (1) in the form

$$\varphi(t) = \varphi'(t) + \varphi''(t) , \tag{36}$$

where

$$\varphi'(t) = \sum_{n=-N}^{N} c_n e^{z_n t}, \qquad \varphi''(t) = \sum_{|n|>N} c_n e^{z_n t}, \qquad (37)$$

and (3) or (34) holds with $\sup\limits_{n}$ replaced by $\sup\limits_{|n|>N}$.

Then all of the above results apply to $\varphi''(t)$. Furthermore, since $\varphi'(t)$ is a finite sum, the series for $\varphi(t)$ and $\varphi''(t)$ have the same convergence properties.

Finally, $ln2$ can be replaced by $2ln2$ in (3). Under this weaker condition, Lemma 3 is valid for $H = L_2(-\frac{1}{2}, \frac{1}{2})$ and $\theta = e^{M/2}-1$. Then the reasoning used in the proofs of Theorems 1 and 2 can be applied to the intervals $(k - \frac{1}{2}, k+\frac{1}{2})$ to derive similar results. Furthermore, the case of (33) and (34) with the right member of (34) doubled can be handled. Since notational complications are involved, we have preferred to deal with the conditions (3) and (34) as they stand.

5. *An Example.* Consider the difference-integral equation

$$\varphi(t) - \varphi(t+1) = \int_{-1}^{1} K(s)\varphi(t-s)\,ds, \qquad t \geqq 0, \qquad (38)$$

where $K(s), -1 \leqq s \leqq 1$, and $\varphi(t), -1 \leqq t < \infty$, are complex and continuous. This equation was investigated by Anselone and Bueckner [1, 2]. The Laplace transform of $\varphi(t), 0 \leqq t < \infty$, is

$$\hat{\varphi}(z) = \frac{A(z)}{\Psi(z)}, \qquad (39)$$

where

$$A(z) = \int_{-1}^{1} e^{-z(s+t)} K(s)\varphi(t)\,dtds - e^z \int_{-1}^{1} e^{-zt}\varphi(t)\,dt, \quad (40)$$

$$\Psi(z) = 1 - e^z - \int_{-1}^{1} e^{-zs} K(s)\,ds. \qquad (41)$$

An argument involving Rouchés theorem proves that: for some m, $\Psi(z)$ has simple zeros z_n, $|n| \geqq m$, such that $z_n - 2\pi ni \to 0$

as $|n| \to \infty$; and $\Psi(z)$ has only a finite number of other zeros z with $Re(z) \geqq \xi$, where ξ is any real number. By means of (4), $\varphi(t)$ can be expressed in the form

$$\varphi(t) = \varphi'(t) + \sum_{|n| \geqq m} c_n e^{z_n t}, \qquad t \geqq 0. \tag{42}$$

If $\varphi(z)$ is regular for $Re(z) \geqq 0$, then $\varphi' \in L_2(0, \infty)$ and, by Theorem 1,

$$\int_s^{s+\tau} |\varphi(t)|^2 \, dt \to 0 \qquad as \qquad s \to \infty,$$

for each fixed $\tau > 0$. For further details, see the references cited.

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