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" WHICH SUBJECTS IN MODERN MATHEMATICS AND WHICH APPLICATIONS IN MODERN MATHEMATICS CAN FIND A PLACE IN PROGRAMS OF SECONDARY SCHOOL INSTRUCTION ? "

by John G. KEMENY

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1. PREFACE

The International Commission on Mathematical Instruction chose the present topic as one to be studied by national subcommissions in the years 1958 to 1962. When I learned that I was to serve as reporter for this topic at the Stockholm Congress, I contacted all the national subcommissions of ICMI, requesting that reports be sent to me when available. I am very pleased to note that I am now in possession of 21 national reports from all over the world. The following is a summary of these 21 reports, with special emphasis on similarities and differences in points of view.

While I am taking every possible precaution to represent views of various nations accurately and fairly, I fully realize that brief reports cannot reproduce accurately many long years of work. May I therefore take this opportunity to apologize to any mathematician who may feel that the following report is either inaccurate or an insufficient presentation of achievement in his own nation.

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2. The process of change

Only a very few countries reported that so far little or no attempt to introduce modern mathematics had taken place. Of course, this small number may not be significant, since my sample is biased: Presumably countries in which absolutely no attempt to modernize mathematics has occurred have not filed reports on this topic.

Of the remaining countries, the vast majority report that the attempts to modernize the curricula have consisted mostly of informal discussions amongst mathematics teachers and a number of highly encouraging experiments by individual teachers. It seems to be a universal experience that attempts to teach selected topics from modern mathematics well, in reasonable quantities, can be highly successful.

I shall discuss in somewhat more detail reports of a few countries where national reform movements have taken place.

France had a head-start over most other countries in that the French secondary school mathematics program was even traditionally unusually strong. The typical secondary school teacher in France had a strong university degree in mathematics which both placed France into a good starting position and made it easier to introduce modern ideas. Reform started with a series of experiments by teachers trying out various topics of modern mathematics in the classroom. This led to the writing of a series of articles and monographs which were widely discussed. Eventually, a number of seminars were formed at which secondary school teachers and college professors together discussed pedagogical problems involved in curriculum reform. France is fortunate enough to have persuaded a number of its very famous mathematicians to give lectures to high school teachers on topics of modern mathematics. This is all the more remarkable, in that all of this work, both on the part of the lecturers and the high school participants, was entirely voluntary, without compensation. All of this effort finally resulted in success: The Ministry of Education gave its official blessing to plans formulated for the modernization of the secondary school curriculum.

There is also a report of some experiments in France with children of a younger age, to present some basic ideas of geometry, number, and sets from a modern point of view.

Curriculum reform in Germany is complicated by two factors. First of all, in the Federal Republic the problem of education is not in the hands of the Federal Government but of the individual States. Therefore, it is very difficult to initiate a national reform. A permanent conference of ministers of education has been established to provide some degree of uniformity in school curricula. A second complicating factor is the existence of three types of gymnasiums in Germany, with quite different attitudes towards the teaching of mathematics. Real reform has been possible primarily in the mathematics-science version of the gymnasium.

On the other hand, the German gymnasium covers a nine-year period and therefore can provide a continuity in mathematical instruction not possible in most other countries. The German report points out a problem common to many nations —that the amount of time allocated to mathematics in the curriculum is severely limited. Therefore, the introduction of modern mathematics cannot be thought of as the addition of new topics to an existing curriculum. Rather, one must find topics within the traditional curriculum which, although they may have been worthwhile, are not from a modern point of view indispensable. Modern ideas are introduced by the replacement of such topics with selected ideas from modern mathematics. On the other hand, one often has an opportunity to supplement these topics for the better students in an " Arbeitsgemeinschaft ", where students voluntarily go deeper into the subject matter. Apparently such informal courses play an important role in the education of mathematics students in Germany. Not only does

Germany propose a new curriculum for high school mathematics, but their report shows evidence of deep thinking on individual topics in this curriculum. A number of extremely useful articles and monographs have been written in Germany, and the reader will find in the appendix of this report a bibliography from the German report.

The status of Italy seems typical of a large number of countries. Two national commissions have studied the problem of modernizing the high school curriculum, and have reported their findings. Italy is now ready to start implementing these recommendations.

In Israel the Ministry of Education has appropriated funds for the writing of experimental textbooks by a group of mathematicians at Hebrew University.

Poland is an example where, although relatively little actual experimentation has been done in the classroom, there has apparently been an immense amount of highly constructive discussion amongst the teachers of mathematics. The Polish report gives every evidence of having had topics discussed both in a wide range and in great depth; and of highly laudable, constructive thought on the part of many mathematicians. The report indicates that these plans have now reached the stage where they hope to try out experiments on a variety of different lines in the classroom.

A most interesting cooperative enterprise is under way in the Scandinavian countries. They have formed a " Scandinavian Committee for the Modernizing of School Mathematics ". This represents a cooperative effort amongst Denmark, Finland, Norway and Sweden to pool their resources, both mathematical and financial, for the improvement of mathematical education. This is made possible not only by the geographic proximity of these countries but by strong similarities amongst their educational systems, as well as traditional ties.

In 1960 the Committee adopted a 5-point program: (1) To survey mathematical needs both for the use of industries and for the needs of universities. (2) The development of new mathematical curricula. (3) The writing of experimental textbooks. So far four monographs have been produced. (4) Plans have been made for extensive testing of these experimental materials. (5) After these tests have been concluded, the Committee is to make official recommendations to the four governments for the adoption of new curricula for secondary education.

The United States has been unusually fortunate in planning its development of modern mathematics curricula. Reforms of early university mathematics education were being planned a decade ago in the United States. These created new demands for the modernization of high school curricula. A Commission on Mathematics was established and worked through the mid-1950s under the chairmanship of Professor A. W. Tucker, Princeton University. While this Commission had no official national standing, its report has been widely read and has been immensely influential. [Copies may be obtained from the Educational Testing Service, Princeton, N.J.]

As soon as this report was published, it became clear that at least two steps had to be taken to make any reform in the United States a reality. One was the introduction of suitable text materials, even if they were of an experimental nature. The second was the training of tens of thousands of high school mathematics teachers who had never been exposed to modern mathematics. Here the National Science Foundation came to the aid of the mathematicians. Through grants, amounting to many millions of dollars, the National Science Foundation established means of meeting both of these problems.

First of all, special institutes were established for the re-training of high school mathematics teachers. Each summer thousands of mathematics teachers are enabled to study modern topics in mathematics with all their expenses paid by the Foundation. More recently, the Foundation has enabled mathematics teachers to return to universities for an additional year's study.

The writing of experimental text materials was started by various university groups, notably one at the University of Illinois. More recently, the National Science Foundation made possible the setting up of a national writing group, the School Mathematics Study Group, under the leadership of Professor E. G. Begle, originally of Yale University and now of Stanford University. Over a period of five years more than 100 mathematicians and mathematics teachers have cooperated in the writing of a series of experimental materials. These have been widely tested throughout the United States and have been rewritten until they form both highly acceptable experimental text materials and will form a basis for future textbooks on the subject. (Information about these materials can be obtained from the School Mathematics Study Group, Stanford University, Stanford, California.)

The problem of implementation is made infinitely more complex in the United States than even that noted in the German report, since the final decision on curricula in most cases is neither in the Federal government's hands, nor in the hands of State governments. The latter usually set minimum standards, but the details of curricula are voted on by each individual community. Therefore, before reform is complete, many thousands of local school boards have to be persuaded of the desirability of modernizing their mathematics curricula. On the other hand, this local control also had its advantages in starting wide-scale experimentation. In many states it would have been impossible to get the State governments to approve the new curricula, because of lack of qualified teachers, but individual cities or towns were able to adopt new topics without waiting for State approval. We therefore find a strange situation in the United States, where one may find hundreds of schools with perhaps the most modern mathematics curricula in the world, and at the same time still find thousands of schools that have not even given any thought to the modernization of high school mathematics teaching.

In conclusion, I would like to reiterate a sentiment contained in the German report, namely that it takes at least a generation to complete a major change in the mathematics curriculum. At the rate mathematics is developing, by the time the present reform is completed, we are sure to want a reform of the " modern curriculum ".

This is perhaps dramatically illustrated in the United States by some exciting experiments carried out in the last three or four years in teaching modern ideas to students in the first six years of school. For example, in the city of Cleveland, a number of suburban school systems adopted School Mathematics Study Group materials, starting with the 7th year of school, and have developed their own materials for the first six years. They are now facing the very serious problem that by the time their students have studied modern mathematics (in an elementary version) for the first six years, they will find the " frightening " new ideas of the 7th and 8th years much too easy, and hence these schools will find the modernized curricula terribly old-fashioned.

3. The new curricula

The most striking feature of the 21 reports is the degree of similarity in the proposals for including new topics of mathematics.

There are four areas of modern mathematics that are recommended by a majority of the reports. These are elementary set theory, an introduction to logic, some topics from modern algebra, and an introduction to probability and statistics. Equally frequent is a mention of the necessity for modernizing the language and conceptual structure of high school mathematics.

Perhaps the most frequently mentioned topic is that of elementary set theory. The concept of a set, as well as the operation of forming unions, intersections, and complements, constitute a common conceptual foundation for all of modern mathematics. It is therefore not surprising that almost all nations favoring any modernization of the high school curriculum have advocated an early introduction to these simple, basic ideas. An attractive feature of this topic is that in a relatively short time a student may be given a feeling of the spirit of modern mathematics without involving him in undue abstraction.

It should, however, be noted that in most cases only an elementary introduction of this topic is recommended. For example, the usual " next " topic in developing set theory is that of cardinality. Only three nations have suggested this as a possible topic for inclusion in the secondary curriculum.

The introduction of elementary symbolic logic may be justified on grounds quite similar to that of the introduction of sets. Indeed, the most elementary structures in the two subjects, Boolean algebra and the propositional calculus, are isomorphic. It is, therefore, not surprising that in several countries these topics are studied more or less simultaneously, exploiting the various possible ways of setting up isomorphisms between the systems.

Of course, logic plays a strange dual role in the mathematics curriculum, in that logical reasoning is an underlying feature of all mathematical arguments, and at the same time modern symbolic logic is an interesting topic in its own right. After many centuries of making free use of logic, without careful examination of its basic principles, the mathematician has turned around and made logic one of the branches of mathematics. It should again be noted that in most cases only very elementary principles of logic have been suggested for study in the high school curriculum.

The status of probability and statistics is entirely different from that of logic and sets. The introduction of these subjects into the high school curriculum is proposed usually on the basis of their inherent attractiveness and importance, rather than their instrumental use in other branches of mathematics. In almost all cases both probability and statistics were advocated, usually closely tied together. I shall follow the convention that under the heading of " probability " a branch of pure mathematics is meant, while " statistics " describes a branch of applied mathematics. If this view is accepted, we must see here both the most widely recommended subject in pure mathematics and the only widely recommended subject in applied mathematics, for inclusion in high school education.

I would like to suggest that the extent to which probability theory is to be taught in high school should be one of the topics of discussion following this report to the Congress. Probability theory recommends itself as a very attractive branch or pure mathematics because it is so easy to give examples, from everyday experience, involving probabilistic computations. Therefore, the student is challenged to combine mathematical rigor and intuition.

However one may consider introducing probability theory from a purely classical point of view, in which one deals with equally likely events and defines probability simply as a ratio of favorable outcomes to total number of outcomes. In this case, probability problems reduce to problems of counting or combinatorics. There is no doubt that such simple combinatorial problems are well within the grasp of the average high school student and, indeed, such topics have long been included in high school algebra courses. In many of the reports sent to me it was not clear whether the probability theory advocated goes beyond such elementary computations.

To capture any of the spirit of modern probability theory, it is necessary to introduce the concept of a measure space and to define probabilities of various events in terms of measures of subsets. While anything like a full treatment of measure theory is much too difficult for high school students, a number of experiments have shown the possibility of doing this for discrete situations, or even more restricted, for finite sets. Since the normal problems familiar to high school students deal only with a finite number of possible outcomes, this formulation of the foundations of probability theory corresponds particularly closely to the students' every-day experience. Recommendations for such a very elementary treatment of probabilistic measure theory are contained in four reports.

While a majority of reports contained a suggestion that some topics from modern algebra should be chosen, there was considerably less agreement as to what this choice should be. Basically, there seems to be a split between the advocates of teaching topics from algebraic systems (groups, rings, and fields) and those who advocate linear algebra. In a few cases, both types of topics were suggested, but usually the lack of time in high school curricula prevents the introduction of a very sizable amount of modern algebra.

It seems to me that the motivation for these two types of topics have many common features. The introduction, on an axiomatic basis, of any modern algebra has the very healthy feature of removing the common misconception that axiomatics is somewhat closely tied with geometry. I recall once having a student who told me that, in his experience, the difference between algebra and geometry was that " in geometry you proved things, while in algebra somebody just told you what to do ". Certainly, this objective can be equally well achieved by introducing as one's basic axiomatic system either that of a group or that of a vector space.

In addition to this, either linear algebra or algebraic systems have the advantage of giving deeper insight into certain structures known to the students for other reasons. Linear algebra, of course, has many applications to geometry, while algebraic structures arise as generalizations of one's experience with numbers.

The usual argument given for the introduction of groups, rings, and fields is that this is the only way one can bring about a true understanding of the nature of our number system. Attempts to prove to the student simple rules, such as those governing the operations with fractions, often fail because both the basic assumptions and the results to be proved are too familiar to the student. However, by moving to an abstract axiomatic system, the student is forced to abandon his intuition and rely on mathematical rigor in his proof.

It may certainly be said, if one wishes to introduce one example of an axiomatic system in modern algebra, that the simplest and most universally useful one is that for a group. It also has the attractive feature that, in addition to being applicable to many groups of numbers well known to the student, one can introduce such simple and interesting examples as the symmetries of a simple geometric object (e.g., a square).

A study of vector spaces, of course, is much more difficult than the study of a simple system such as a group. I have not seen any suggestion of studying vector spaces over an arbitrary field. However, there were a number of suggestions for studying a vector space over the real numbers. Here much of the difficulty is removed by relying on the student's intuitive understanding of the underlying field. Presumably, the major motivation for this line of inquiry is that it helps to clarify much of what the student was forced to learn before. For example, it can be used to give new insight into the meaning of the solutions of simultaneous equations. Equally important, of course, are the numerous applications of linear algebra to geometry. While geometry can be used to motivate linear algebra, linear algebra, in turn can be used to make the nature of geometric transformations more clearly understood.

I must now mention a few topics which occurred occasionally amongst the recommendations, though these seem to be topics not nearly so widely accepted. These include some modern topics in geometry, the study of equivalence and order relations, cardinal numbers, and an introduction to elementary topology. There were also scattered mentions of applications, but this is a topic to which I wish to return later.

There seems to be general agreement that the teaching of high school geometry must be modernized, but there is a certain lack of ideas as to how this should be achieved. I recall the detailed debate at the 1958 International Congress on this particular topic, and I am under the impression that this problem is still far from settled.

For example, the School Mathematics Study Group in the United States wrote single textbooks for each of six years for junior high school and high school mathematics. However, in the case of the tenth year, there are already two different versions of geometry available, and there may very well be a third version. This is a clear-cut indication of the lack of agreement amongst leading mathematicians in the United States as to the " right " way of teaching geometry.

The most constructive suggestions on this topic seem to be contained in the report from Germany, and I refer the reader to the excellent bibliography contained in the appendix. I share the astonishment expressed by the German reporter that high school geometry has remained so terribly tradition-bound, even in the face of many changes in the teaching of algebra, and the introduction of more advanced topics. We must choose between a 2,000-year-old tradition of teaching synthetic geometry in the manner of Euclid, or of destroying the " purity " of geometry by the introduction of algebraic ideas. Of course, Felix Klein established a very important trend in Germany, which spread throughout the world, to attempt to build a classification of geometries by means of the transformations which leave certain geometric properties invariant. This points to the importance of the study of geometric transformations, even within high school geometry. There is also an increasing tendency to introduce metric ideas early into synthetic geometry and in many countries even an introduction to analytic geometry is part of the first year's geometry course.

The introduction of vectors is quite generally advocated. In Germany vectors are introduced in the context of metric (as opposed to affine) geometry. However, this does not mean that vectors are tied to analytic geometry, since vector methods are used as a substitute for the introduction of a coordinate system. This approach is particularly useful in bringing out the analogy between the geometries of two, three, and more dimensions.

A conference sponsored by ICMI at Aarhus, in Denmark, in 1960, advocated the development of a " pure " vector geometry, in which affine geometry is built up in terms of vector ideas. While the concept of vectors free of coordinate systems may be somewhat more difficult for the beginning student to understand, many geometric proofs actually become much simpler if vectors are treated as coordinate-free. For example, this is by far the easiest way to prove that the medians of a triangle meet at one point and divide each other in a 2:1 ratio.

While there are still many advocates of treating a full axiomatic system of Euclidean geometry purely synthetically, it is becoming increasingly clear that one must either " cheat " or demand more of the student than can be expected of him in his high school years. Even Euclid's original axiom system is a great deal more complex than is ideal for the high school student's first introduction to axiomatic mathematics. In addition, it is well known that Euclid in many places substituted intuition or the drawing of a diagram for mathematical rigor. Indeed, many of Euclid's propositions do not follow from his axioms. While several outstandingly fine axiom systems have been constructed that make Euclidean synthetic geometry rigorous (notably the system by Hilbert), these require a degree of mathematical maturity not to be expected of the secondary school student.

The report from Israel feels that the axiomatic treatment of geometry in high school is as unrealistic as using Peano's postulates in elementary school. The report from the United States, in contrast, advocates that certain *segments* of Euclidean geometry be taught rigorously, to give the student experience in proving theorems from axioms, but that the gaps in between be filled in by a more intuitive presentation, in which the emphasis should be in teaching students the " facts of geometry ". An alternative to this is the much heavier reliance on the properties of real numbers to fill in gaps in Euclid's axiom system.

Three reports advocated the inclusion of non-Euclidean geometry as part of the first treatment of Euclid. The argument for this is similar to the argument for teaching algebraic systems to improve the students' understanding of number systems. That is, if the student is forced to reason in a geometric framework other than the one he is used to, he is more likely to understand the power of the deductive system and to appreciate proofs he has seen in Euclidean geometry. I should like to add a plea that, even in courses where no actual non-Euclidean geometry is taught, the student should at least be informed that such geometries do exist, and perhaps a day or two be spent discussing them. It seems to me to be a major cultural crime of most mathematical educational systems that 130 years after the invention on non-Euclidean geometry, most students (and many teachers) are not aware of the possibility of a non-Euclidean geometry. Indeed, the statement that our universe is only approximately Euclidean, according to relativity theory — it may both in the small and the large be non-Euclidean — comes as a great shock to many pedagogues.

A frequently mentioned topic is a brief study of relations in general, with special emphasis on equivalence relations and order relations. The justification for such fundamental concepts is the same as for a brief study of sets and of symbolic logic; once these concepts are introduced, they can be used again and again to clarify later topics.

Three reports suggested the inclusion of a systematic study of cardinal numbers. I must say that this suggestion both delights me and surprises me. It delights me in that I have always been critical of university education in the United States, in that most students are supposed to learn the facts about infinite cardinals entirely on their own, since these topics are rarely explicitly taught in courses. The suggestion surprised me because I had felt that this topic was too difficult for high school curricula. If various countries succeed in this experiment, I think it would be most useful if the results were widely publicized.

Suggestions of a brief introduction to topology are contained in four reports. The French report proposes that an intuitive notion of neighborhoods be given to students and on this one should base the concepts of the convergence of a sequence (or the failure of convergence) and that these ideas should be used to lead in a natural way to the concepts of limits and continuity. These can in turn be used to explain such geometric ideas as that of a tangent or of an asymptote. Germany and Israel make similar suggestions.

A more ambitious program is outlined in the Polish report. The proposal is that most of the treatment be restricted to the topology of Euclidean space of one, two, and three dimensions. Starting with these well-known spaces, the concept of a metric space should be developed, and, in turn, illustrated on such examples as n-dimensional space, the space of continuous functions, and Hilbert space. The Polish program would start with the same concepts as mentioned above from the French report. However, by limiting itself to more concrete examples, it proposes to go considerably further. Such general concepts as connectedness, boundary, homeomorphism, and continuous mappings would be discussed. More concretely, it is suggested that discussions without proofs should be given of the Jordan-curve theorem, classification of polyhedral surfaces, and some examples of non-orientability of surfaces. The unit would terminate with a discussion of Euler's theorem.

Young reporter would like to add his support to this suggestion, even though it may sound quite extreme. While these topics may be too difficult for the average high school student, I know from personal experience that the really bright student, in his last year of high school, is fascinated by elementary topological ideas. Such a unit should be entirely practical as long as it is closely tied to concrete examples familiar to the student.

Most of the reports contained frequent mentions of traditional topics whose teaching would be improved by the adoption of a more modern point of view. As one example, I shall use a unit discussed in the report from the United States. This is the treatment of equations, simultaneous equations and inequalities. An equation or inequality is treated as an " open sentence ". That is, it is a mathematical assertion which in itself is neither true nor false, but becomes true or false when its variables are replaced by names of numbers or points (or more abstract objects, in advanced subjects). Therefore, the solution of an equation is the search for the set for which the assertion is true. This set is commonly referred to as the " truth set " or the " solution set ".

Thinking of solutions of equations as sets has the advantage that a student is more likely to think of the possibilities of the solution having more than one element in it or, for that matter, being the empty set. Simultaneous equations may be thought of as conjunctions of several open sentences; hence their solution consists of the intersection of the individual truth sets. This point of view makes it much easier to explain the usual algorithms for solving of simultaneous equations. The attempt in any such algorithm is to replace a set of sentences by an equivalent set, i.e., one having the same truth set, but the latter being of a form in which the nature of the solutions is obvious. The approach also has the advantage that equations and inequalities may be treated in exactly the same manner. The graphing of equations and inequalities, then, simply becomes a matter of graphical representation of truth sets. In this case, the meaning of " intersection " of solution sets becomes particularly clear.

4. Applications of mathematics

It is painfully clear, in reading the 21 national reports, that relatively little attention has been given by our reformers to the teaching of applications of mathematics. The only notable exception to this is the inclusion of statistics in a majority of the recommendations. Aside from this, only scattered suggestions are made, none of them occurring in more than two reports. Indeed, some reporters have specifically complained that, while an enormous effort has been made in their nations to improve the teaching of pure mathematics, the topic of applied mathematics has apparently been forgotten. I would like to propose to ICMI that a study of the teaching of applications of mathematics should receive high priority in its studies of the next four-year period.

Aside from statistics, three types of applications have been mentioned. One is applications of mathematics to physics. I presume it differs greatly from country to country as to whether topics such as mechanics are included in the mathematics curriculum or are treated in separate physics courses.

A second area that was mentioned twice was that there are great possibilities in the future of improving the teaching of mathematics by making free use of computing machines. Of course, in the immediate future this may not be practical until highspeed computers are available in large enough numbers for high school students to be able to give sufficient time on them.

A third area mentioned was linear programming. This particular topic has the attraction that it ties up nicely with linear algebra and therefore can reinforce the teaching of a quite modern topic of abstract mathematics. It also lends itself to good numerical problems which are both interesting and will exercise the student's ability in the solving of equations. But, above all, it may be the only example the student will see of a genuine application to the social sciences.

The philosophy of teaching applied mathematics is particularly well described in the report from the Netherlands.

" It is an urgent problem whether secondary education must restrict itself to pure mathematics. Applications gain more and more momentum in the social system. If these applications were only operational, one could ask whether they should be taught at all in high schools. Teaching applied mathematics, however, implies developing new habits of thinking, which in many cases differ from those in abstract mathematics. For instance, in statistics it is difficult to acquire operational skill as long as one has not really and independently understood the fundamental notions. "

5. Further observations

Perhaps the major motivation for teaching modern mathematics, or mathematics in a modern spirit in high school, is to prepare the student for his university experience. The need for this is particularly well brought out in a quotation from the French report from Professor Lichnerowicz. The quotation (in translation) reads: " The classical teaching of our lycées in a large measure conditions our students to a certain conception of mathematics, a conception which is ... derived from the Greeks, and ... from the experience of mathematicians of the middle of the nineteenth century ... At the university, the students suddenly encounter the spirit of contemporary mathematics, a painful shock ... The student must totally ' recondition ' himself ... and this is translated by an expression which I personally have often heard: ' What you are teaching is no longer mathematics ' ... "

I am sure that many of us can testify to the same experience. Let us now examine a few pedagogical problems.

The Netherlands report recommends that " stress should be laid on thinking mathematically and more value attached to this ability than to knowledge of a variety of less important facts. " If this philosophy is adopted, then presumably the exact choice of topics is not nearly as significant as the manner in which they are presented in the high school.

An important pedagogical idea is expressed in the Portugese report: " For this introduction (of modern mathematics) it would be essential to bring out many concrete examples, well known and quite suggestive, as well as amusing, and one would be careful not to introduce formalism until one was sure that the student had grasped the ideas behind them. "

One question that arises in the introduction of new topics is what topics are reduced to make room for the inclusion of new ideas. By far the most frequently mentioned topics were a reduction in the amount of time spent on synthetic geometry, a considerable reduction of trigonometry, especially the emphasis on triangle solving, a reduction of solid geometry, possibly by incorporating it into the first course in geometry, and a reduction in some of the traditional and not very practical numerical methods included in algebra courses.

A pedagogical question on which there seems to be considerable disagreement is the extent to which high school mathematics should be axiomatized. I found several recommendations that there should be a substantial extension of the body of axiomatics in high school, or even that axiomatic systems, as such, should be studied. On the other hand, there were about an equal number of objections to excessive use of axiomatization in the modernized curricula. For example, " The enrichment of the syllabus by the insertion of interesting examples of modern elements of mathematics is to be encouraged, and indeed is bound to happen. But the systematising of teaching in line with axiomatic mathematical theories would lead to a situation contrary to accepted British teaching principles. "

A different view concerning axiomatics is shown in the French report: " *Axiomatic Exposition.* A program cannot demand that teaching have an axiomatic character until sufficient scientific experience permits the student to feel its need. Axiomatic procedure is extremely rigid, each step is strictly controlled, appeal to the intuition has no value because the choice of axioms accepts some facts and rejects others just as sympathetic to our intuition. If the construction succeeds and gives what our experience of the question expected, if one has more or less demonstrated the independence of the axioms and the categorical quality of their set, one sees that the choice was good. But who will believe that such a choice can be made without fumbling ? and different axiomatizations are valid. It is impossible to set them forth without dogmatism, without appealing to the authority of the teacher who is able to show only to the end that the work is valid.

" In secondary school teaching one can only try to come to the conclusion that axiomatics are doubtless possible and desirable in mathematics. In terminal classes, it is recommended to do a few axiomatic expositions at the outset, granting the necessity of afterward accepting a more technical viewpoint. But it is very dangerous to do partial axiomatics, which hide the unity of mathematics even if one doesn't make vicious circles (like using number to axiomatize geometry and geometry to axiomatize the notion of number !).

" However, even if a large place is left to the intuition of the children and the path chosen for exploring the program is flexible and takes account of the spontaneity of the students . . . it is necessary for the teacher to impose an order without which there would be only confusion. This order reflects an underlying axiomatization adopted by the teacher, of which the best pupils can become aware at the end of the school year. "

In the historical development of mathematics, it is usually, though by no means always, the case that a certain body of mathematical facts is first discovered, and then one or more people perform the very important task of systematizing this information by specifying a minimal number of axioms and deriving the other facts from these. It is therefore clear both that some acquaintance with axiomatic mathematical systems is an important part of mathematical education, but also that mathematics is something over and above mere development of axioms. Just what the happy compromise is between these two trends may be a topic well worth discussing at the Congress.

The newly developed Danish curriculum provides a very interesting idea — namely, an optional topic to be selected by the high school teacher. The choice of this topic is described as follows:

" Contents, extent and mode of treating the optional subject should be adapted in such a way that the students are not in this field faced with more difficult problems than those arising from the other lessons of mathematics.

" Some examples of the fields from which the optional subjects may be taken: History of mathematics, number theory, matrices and determinants, theory of groups, set theory, Boolean algebra, differential equations, series, probability theory, statistics, theory of games, topology, projective geometry, theory of conics, noneuclidian geometry, geometry of higher dimensions, geometrical constructions, descriptive geometry.

" The optional subject may also be chosen in connection with the corresponding part of the physics course. As examples of suitable subjects may be mentioned: Probability theory and kinetic theory of gases, differential equations and oscillatory circuits. Finally the optional subject may be organized in connection with other subjects than physics, e.g., probability theory and heredity.

" The program for the optional subject will have to be submotted to the inspector of schools for approval.

" The existence of an optional subject in the mathematics curriculum is new in Denmark. This subject will have such an extent that a couple of months in grades 11 or 12 will be occupied by it. Of course, both modern and classical subjects will be chosen, but it is expected that many teachers will choose the theory of probability as their teaching subject. In the list of non-optional subjects probability does occur, but only on a very modest scale. Of course the teacher is free to choose between an axiomatic and a non-axiomatic treatment of probability, but certainly an axiomatic treatment will be used by some teachers. (This will probably be easier to carry through if one restricts oneself to discrete sample spaces.) In this case the pupils will get a very useful impression of a simple axiom system and an example of a mathematical model. "

One topic mentioned in a number of reports is the extent to which calculus is included in the secondary school curriculum. I have not specifically discussed this topic since it cannot legitimately come under the heading of " modern mathematics ". However, it is clear that there are increasing pressures from physical scientists to teach some units in calculus in our secondary school curricula, and to a great extent this pressure may compete with the demands for modernizing of modern mathematics. Let me simply indicate that at the present time there are vast differences from the majority of countries that teach no calculus at all in the secondary school to the large number of countries that teach a first, more or less intuitive introduction to calculus, to such extreme examples as the recent experiment in Sweden. A special experimental unit will be taught in that country on differential equations: " This small course consists of linear equations of first order and of second order, with constant coefficients. Proofs of existence and uniqueness are given. "

The Hungarian report calls attention to two problems that have caused difficulties in modernizing the high school curriculum: " One is the preparation of the teachers now teaching for the handling of new subjects. Without this, the introduction of such topics cannot succeed. But equally important is the formation of public sentiment, since for the majority of people it is not obvious why their children in high school should learn about problems that their parents may never have heard of in their entire lives. We have to solve these problems simultaneously with the modernization of the curriculum. "

I am quite certain that many reporters would heartily support these remarks. There are indications in many reports that major national attempts have been made to modernize the training of existing high school teachers. This is, of course, often a highly painful and difficult experience for adults who have left their universities with the impression that they are prepared to teach mathematics for the rest of their lives, and find themselves forced to return to study what often seems to them strange new ideas.

Speaking for the United States, I may add that the problem of informing parents of high school children is equally critical. In many communities where the schools were happy to modernize the mathematics curricula they ran into unexpected opposition from parents who simply could not understand why modern mathematics should be taught, or even how there could possibly be such a thing as modern mathematics. It is strange that, in an age of fantastically rapid development in mathematical research, perhaps a majority of laymen are under the impression that all new mathematics was done hundreds of years ago.

Most of the reports were from countries with educational systems based on centuries of tradition. I was fortunate in obtaining one report from Africa, which painted a fascinating picture of the problems faced by newly developing nations. I would like to reproduce just one quotation which I found particularly interesting, from the report of Sierra Leone:

" The most important factor in our survey is that in all these areas education has been expanding very, very rapidly within the last ten years. The number of secondary schools has at least doubled in all areas and is still expanding. It is in these new schools that there is the greatest opportunity for introducing modern mathematics. The teachers in these schools are usually young enthusiasts and, the schools often being in new towns, are sufficiently separated from the older traditional schools to make it possible for experimental work to be carried out without pupils and parents continually comparing the work there with the work being done in other schools. "

6. Conclusions

It is clear from the reports that many nations have made an excellent start on the modernization of high school mathematics curricula. It is equally clear that much hard work still needs to be done.

There seems to be fairly general agreement that some basic concepts from set theory and logic should be introduced, that geometry should be modernized, that some elements of modern algebra be introduced, and that probability and statistics are suitable for high school teaching. Even more important is the general agreement that much of traditional mathematics should be taught from a modern point of view. However, as far as the details of these recommendations are concerned, there is considerable disagreement.

The two greatest difficulties blocking progress are the critical shortage of qualified teachers, and the lack of suitable text materials. The former problem has been attacked in a few countries by running special courses for high school teachers whose training was mostly traditional. The latter is being solved by the writing of many excellent experimental text materials.

I should like to conclude the report by making two specific recommendations to ICMI:

Recommendation 1. That ICMI initiate study on three problems that have arisen out of the national reports: (1) How can the teaching of applied mathematics in our high schools be modernized ? It is clear that this problem has been neglected in the past. (2) To what degree should high school mathematics be axiomatized ? There is considerable disagreement on this topic. (3) How and to what degree should probability theory be introduced ? While this is the subject most frequently recommended as a major new topic, many pedagogical questions concerning it remain to be answered.

Recommendation 2. That ICMI serve as a clearing house for experimental materials on modernizing high school mathematics. That each national subcommission should be requested to send to ICMI a list of available books and articles, with an indication of how they can be obtained, and that this list be kept up to date by ICMI and circulated to the national commissions. This could expedite planning and eliminate unnecessary duplication.

APPENDIX

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Le samedi 8 février 1964, à l'Institut Henri Poincaré, le Comité Albert CHÂTELET a remis en présence de la famille CHÂTELET et de M. Marc ZAMANSKY, doyen de la Faculté des Sciences, la médaille Albert CHÂTELET 1963 à Mme Yvette AMICE, ancienne élève de l'Ecole Normale supérieure des jeunes filles, pour ses travaux d'analyse p-adique.