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Artikel: ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS

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for all u, v in the shell

$$R \leqq \| u - u_0 \| \leqq \frac{k(c - mR)}{1 - km}.$$

b) Let $\| (I - K)^{-1} \| = k$. There exist numbers R and $m < k^{-1}$ such that

$$\| (W - K)v - (W - K)u \| \leqq m \| u - v \| \text{ if } \| u \| > R \text{ and } \| v \| > R.$$

12. Non-linear equations containing a linear completely continuous symmetric operator.

As we have seen in some previous theorems, under certain general conditions, the existence of a solution of an approximating equation or the existence of a solution at all, can fail only if there is no approximating linear operator with bounded inverse or if there is not everywhere such an operator. In the cases when the operators considered are differentiable this means that the derived linear operator does not have a bounded inverse or the derived linear equation fails to have a unique and bounded solution.[1]) It is, therefore, important to have conditions for the existence of a bounded inverse of a corresponding linear operator.

In the case of an operator $I - A$, where A is completely continuous, this is equivalent[2]) to the fact that $u = Au$ has only the solution $u = \theta$, i.e. 1 is not an eigenvalue of A. Here we deal only with such cases and assume our non-linear equation to have the form

$$u = LVu, \tag{12.1}$$

where L is a completely continuous operator and V is an (in general non-linear) operator. This is, indeed, the most usual form of non-linear equations with a completely continuous operator.

Moreover, we now consider the equation (12.1) in a Hilbert space, that is, the operator LV has its domain and range in a

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1) This is, of course, typical for the " regular case " of non-linear equations.
2) See footnote 2 on page 47.

Hilbert space H. Finally, throughout this section, let L be a symmetric operator.

Under these general assumptions we will give conditions that the derived equation

$$v = LV'_{(u)}v, \tag{12.2}$$

have only the trivial solution, $u = \theta$.

To this end we first note some well known statements[1]) on the eigenvalues of a completely continuous symmetric operator: Let A be such an operator defined on a Hilbert space H and with range in H, A being different from the zero-operator.

Then there exists a finite or infinite orthonormal set[2]) of eigenvectors e_i corresponding to real eigenvalues λ_i such that every $u \in H$ can be written uniquely in the form

$$u = \sum_i a_i e_i + u' \quad \text{where} \quad Au' = \theta. \tag{12.3}$$

Let us arrange the sequence of eigenvalues as follows:

$$\lambda_{-1} \leqq \lambda_{-2} \leqq \dots\dots \leqq \lambda_2 \leqq \lambda_1, \tag{12.4}$$

where the λ_n (λ_{-n}), $n \geqq 1$, are positive (negative). One of the two sequences may be empty.

Together with $Au = \lambda u$ we consider the equation

$$u = \kappa A u, \quad u \neq \theta. \tag{12.5}$$

Then, we have the corresponding sequence[3])

$$\dots \leqq \kappa_{-2} \leqq \kappa_{-1} < 0 < \kappa_1 \leqq \kappa_2 \leqq \dots, \tag{12.6}$$

of " characteristic values " $k_i = \dfrac{1}{\lambda_i}$ instead of (12.4).

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1) See, for example, F. Riesz and B. Sz.-Nagy [19], chapter VI, and A. N. Kolmogorov and S. V. Fomin [18], II, section 27.

2) $Ae_i = \lambda_i e_i$, $(e_i, e_k) = \delta_{ik}$.

3) The terminology differs in the literature. We define the " eigenvalues " according to the previous sections by $Au = \lambda u$, $u \neq \theta$.

By means of the maximum-minimum principle[1]) we have the independent representations

$$\lambda_1 = \sup_u \{ (Au , u) : \| u \| = 1 \} \qquad \text{and}$$
$$\lambda_n = \inf_{v_i} \sup_u \{ (Au , u) : \| u \| = 1 , (u , v_i) = 0, i = 1, \dots , n-1 \} \tag{12.7}$$

if λ_1 and λ_n, respectively, exist, that is, if the expressions on the right hand side are positive. For λ_{-1} and λ_{-n} we have analogous representations, but the supremum and the infinum must be interchanged.

We now introduce the set P of operators, $p \in P$, which have the following properties:

a) $p \in P$, $u \in H$ implies pu exists and $pu \in H$.

b) All $p \in P$ are linear, continuous, and symmetric,

c) (pu, u) is real for all $u \in H$.

If α is a real number, we write $p < \alpha$, $p \leqq \alpha$, $p > \alpha$, $p \geqq \alpha$ when the corresponding product (pu, u) is $<$, $\leqq$, $>$, $\geqq \alpha (u, u)$, respectively, for all $u \in H$, $u \neq \theta$.

d) If $p \in P$, $p \geqq 0$, then $\sqrt{p} \in P$, $(\sqrt{p})^2 = p$, and $\sqrt{p} < 0 \, (\geqq 0)$ when $p > 0 \, (\geqq 0)$.

Then, obviously, all real numbers α belong to P. It is easy to show that with A and $p \geqq 0$ also the operator $C = \sqrt{p}\, A \sqrt{p}$ is linear, completely continuous, and symmetric. Furthermore, if $p > 0$, then $\sqrt{p} u = \theta$ implies $u = \theta$ and the eigenvalues of Ap and those of $\sqrt{p}\, A \sqrt{p}$ coincide. In fact, $Ap\varphi = \lambda\varphi$ and $\varphi \neq \theta$ imply $\sqrt{p}\, A \sqrt{p}\, \Psi = \lambda \Psi$ with $\Psi = \sqrt{p}\varphi \neq \theta$. The operator $\sqrt{p}\, A \sqrt{p}$ is self-adjoint if A is self-adjoint and $p \geqq 0$, $p \in P$. Therefore, the eigenvalues of Ap are real. On the other hand, if $p > 0$ and $\sqrt{p}\, A \sqrt{p}\, \Psi = \lambda \Psi$ then $\sqrt{p}^{-1}$ exists because $\sqrt{p} u = \theta$ implies $u = \theta$ and with $\varphi = \sqrt{p}^{-1} \Psi$ we have $\sqrt{p}\, Ap\varphi = \lambda \sqrt{p}\varphi$ which implies $Ap\varphi = \lambda\varphi$. We have the development

$$u = \sum_i c_i \Psi_i + u' \quad \text{where} \quad \sqrt{p} A \sqrt{p} u' = \theta ,$$

1) Courant-Hilbert [20], chapter III, § 3.

and $\{ \Psi_i \}$ is a set of orthonormal eigenvectors of the self-adjoint operator $C = \sqrt{p}\, A \sqrt{p}$.

After these considerations we can prove the following theorem.[1])

THEOREM 12.1. Let A be a linear completely continuous symmetric operator on a Hilbert space H into H, let κ_i be its characteristic values (according to (12.5), (12.6)), and let $p \in P$.

Then the equation

$$u = Apu\,, \tag{12.8}$$

has only the solution $u = \theta$, i.e., $\mu = 1$ is not an eigenvalue of Ap, if one of the following conditions holds:

a) κ_n and κ_{n+1} (κ_{-n} and $\kappa_{-(n+1)}$), $n \geqq 1$ exist and $\kappa_n < p < \kappa_{n+1}$ ($\kappa_{-n} > p > \kappa_{-(n+1)}$) .

b) κ_n (κ_{-n}) exists as the largest positive (smallest negative) characteristic value and $p > \kappa_n$ ($p < \kappa_{-n}$) .

c) There is no positive (negative) characteristic value and $p \geqq 0$ ($p \leqq 0$) .

d) κ_1 (κ_{-1}) exists and $0 \leqq p < \kappa_1$ ($\kappa_{-1} < p \leqq 0$) .

e) $\| p \| < \min\limits_i (| \kappa_i |)$.

Proof. a_1) Let the *n-th* positive characteristic value κ_n of A exist and let $p > \kappa_n > 0$. We show that then the *n-th* positive eigenvalue μ_n of $C = \sqrt{p}\, A \sqrt{p}$ is greater than 1.

Let $\{ e_i \}$ and $\{ \Psi_i \}$ be the sequences of orthogonal and normed eigenvectors of the operators A and C, respectively, corresponding to the eigenvalues $\{ \lambda_i \}$ and $\{ \mu_i \}$, respectively.

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1) In the special case of the boundary value problem $(g(x)y')' + p(x)y = 0$, $y(x_2) = 0$, $y(x_1) = 0$, most of the results follow easily from the Sturm comparison theorem. See, for example, E.A Coddington and N. Levinson [21], chapter 8. In some cases of special equations in which stronger conditions such as $\kappa_n < \alpha_n \leqq p \leqq \alpha_{n+1} < \kappa_{n+1}$ instead of a) hold, the results can be obtained from other well known comparison theorems for eigenvalues, appearing, for instance, in L. Collatz [10], § 9, and F. Riesz and B. Sz.-Nagy [19], section 95.

The system

$$u = \sum_{v=1}^{n} c_v e_v , \quad \| u \| = 1 , \quad (u , \varphi_i) = 0 , \quad i = 1, \ldots, n-1 ,$$

with $\varphi_i = \sqrt{p}^{-1} \Psi_i$, that is

$$\sum_{v=1}^{n} | c_v |^2 = 1 , \quad \sum_{v=1}^{n} c_v (e_v , \varphi_i) = 0 , \quad i = 1, \ldots, n-1 ,$$

is always solvable. For such *a* u, by (12.4), we have

$$(Au , u) = \sum_{v=1}^{n} \lambda_v | c_v |^2 \geqq \lambda_n .$$

Hence

$$\lambda_n \leqq \sup_u \{ (Au , u) : u = \sum_{v=1}^{n} c_v e_v , \| u \| = 1 , (u , \varphi_i) = 0, \quad i = 1, \ldots , n-1 \} \tag{12.9}$$

$$\leqq \sup_{v \neq \theta} \left\{ \frac{(A\sqrt{p}v , \sqrt{p}v)}{(\sqrt{p}v , \sqrt{p}v)} : (v , \Psi_i) = 0 , \quad i = 1, \ldots , n-1 \right\}$$

since $(\sqrt{p}v, \varphi_i) = (v, \Psi_i)$, $i = 1, \ldots , n-1$, and the first supremum on the right hand side can only become larger if we drop the condition

$$u = \sum_{v=1}^{n} c_v e_v .$$

The assumption $p > \kappa_n = \dfrac{1}{\lambda_n} > 0$ yields

$$\frac{(A\sqrt{p}v , \sqrt{p}v)}{(\sqrt{p}v , \sqrt{p}v)} = \frac{(Cv , v)}{(pv , v)} < \lambda_n \frac{(Cv , v)}{(v , v)} . \tag{12.10}$$

Since the bounded set $\{ c_v \}$ satisfying (12.9) is compact the supremum in (12.9) is actually assumed. Therefore, from (12.9) and (12.10) we get

$$\lambda_n < \lambda_n \sup_v \{ (Cv , v) : \| v \| = 1 , \ (v , \Psi_i) = 0 , \ i = 1, \ldots , n-1 \} = \lambda_n \mu_n \quad \text{or} \quad \mu_n > 1 .$$

a_2) If κ_{n+1} exists and $0 < p < \kappa_{n+1}$ we obtain $\mu_{n+1} < 1$ by a similar argument where the roles of A and C as well as the roles of λ and μ are interchanged.

Thus the equation (12.8) does not have the eigenvalue 1, that is, the theorem holds true for the case *a)* with positive $p \in P$.

b) If κ_{n+1} does not exist but κ_n does, i.e., the right hand side of (12.7) is positive for n but not positive for $n+1$, then, replacing u by $\sqrt{p}u$ with $\kappa_n < p$, we obtain that

$$\inf_{v_i} \sup_{u} \{ (\sqrt{p}A\sqrt{p}u, u) : \| u \| = 1 , \quad (u , v_i) = 0 , \quad i = 1, \dots , n \}$$

also cannot be positive, i.e., $\mu_{n+1} > 0$ does not exist either. From a_1*)* it follows that in this case $\mu_n > 1$ is the smallest positive eigenvalue, i.e. the theorem holds for the case *b)* with positive κ_n and p.

c) If there is no positive eigenvalue then $(Au, u) \leqq 0$ for all u, which obviously implies $(\sqrt{p}\, A \,\sqrt{p}u, u) = (A \sqrt{p}u, \sqrt{p}u) \leqq 0$ for $p \geqq 0$. Thus 1 is not an eigenvalue.

d) In this case the proof is similar to a_1*)* and a_2*)* if $p \geqq 0$: the largest eigenvalue μ_1 becomes less than one here.

The cases of negative eigenvalues and negative *p's* can be easily reduced to the positive cases treated above. Let λ_ν^- and k_ν^- be the eigenvalues and characteristic values, respectively, of the operator $-A$. So we have $\lambda_{-n} = -\lambda_n^-$ and the same with κ_ν^-. From $\kappa_{-(n+1)} < p < \kappa_{-n}$ it follows that $\kappa_{n+1}^- > -p > k_n^-$. Because $Ap = -A\,(-p)$ we can, therefore apply the above results to $-A$ and $-p$ instead of A and p, respectively.

e) We have [1])

$$\min(|\kappa_i|) = \min\left(\frac{1}{|\lambda_i|}\right) = \frac{1}{\max(|\lambda_i|)} = \| A \|^{-1} .$$

Therefore, it follows under the condition *e)* that

$$\| Ap \| \leqq \| A \| \cdot \| p \| < 1 .$$

Hence, 1 is not an eigenvalue.

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[1]) See, for example, N. I. Achieser and I. M. Glasmann [14], p. 47.

This completes the proof.

Theorem 12.1 can be applied to all previous theorems which use the fact that the derived linear equation has only the zero-solution to establish the solvability of the given non-linear equation, provided that this equation can be written in the form

$$u = LVu, \tag{12.11}$$

with a linear, completely continuous, and symmetric operator L. In these cases we are able to give explicit conditions on the derivative $V'_{(u)}$ of V as essential conditions for the existence of a solution of (12.11). This derivative plays the part of the operator $p \in P$ in Theorem 12.1. We remember that, in this sense, $V'_{(u)} > \kappa$ is equivalent to $(V'_{(u)}\varphi, \varphi) > \kappa(\varphi, \varphi)$ for all $\varphi \in H$, $\varphi \neq \theta$, and the same with $\geqq$, $<$, and $\leqq$. We now give a few examples, first a neighborhood theorem:

Theorem 12.2. Let the product operator LV with a linear completely continuous symmetric operator L and a non-linear continuously differentiable operator V be defined on a Hilbert space H and have its range in H. Let $V'_{(u)}$, $u \in H$, satisfy one of the conditions *a)* through *e)* of Theorem 12.1 with $A = L$ and $V'_{(u)} = p \in P$.

Then for each point $(u_0, w_0 = u_0 - LVu_0)$ there exists an $\Omega = (u_0, r, a, b)$-neighborhood in which the equation

$$u = Tu + w, \quad (w + I - T \in \Omega),$$

is uniquely and continuously solvable. In particular, the equation

$$u = LVu + w, \tag{12.12}$$

has a unique and continuous solution $u(w)$ for w and u in certain spheres about w_0, u_0, respectively, i.e., $I - LV$ has a local inverse there.

The proof follows from Theorem 7.1 and supplements and the fact that a completely continuous operator has only a point spectrum. Therefore, the operator $(I - LV'_{(u)})^{-1}$ is bounded under the assumptions of Theorem 12.2.

The conditions of this theorem are not sufficient for the existence of a solution of (12.12) for each $w \in H$ or, in particular, for $w = \theta$. But as in previous sections, simple additional assumptions assure the existence of a solution of (12.12) for an arbitrary given $w \in H$.

THEOREM 12.3. Let L and V satisfy the conditions of Theorem 12.2 and let one of the following assumptions be fulfilled:

a) For some $u_0 \in H$ and $w_0 = u_0 - LVu_0$ let the set

$$U = \{u : u = LVu + w_0 + \lambda(w - w_0)\,,\ 0 \leqq \lambda < 1\} \qquad (12.13)$$

be bounded.

b) For some $u_0 \in H$ and $w_0 = u_0 - LVu_0$ let the set

$$S = \{s : s = \| k \| \cdot \| (I - LV'_{(u)})\, k \|^{-1},\ k \in H,\ u \in U\}, \qquad (12.14)$$

where U is defined in (12.13), be bounded.

Then the equation (12.12) has a solution.

For the proof we set

$$T_\lambda u = (I - LV)\,u + w_0 + \lambda(w - w_0)\,, \quad 0 \leqq \lambda \leqq 1\,,$$

and denote by Λ the set of all λ in $[0,\ 1]$ for which $T_\lambda u = \theta$ is solvable. Λ is non-empty because $\lambda = 0$ belongs to Λ. Theorem 12.2 proves Λ is open with respect to $[0,\ 1]$. Λ is also closed. This can be shown in the case *a)* in the same way as in the proof of Theorem 10.3 under λ) where the operator V is to be replaced by LV, and in the case *b)* the proof follows from Theorem 9.1 with $Tu = (I - LV)\,u + w$ and $T_0\,u = (I - LV)\,u + w_0$.

As already remarked in section 9 before corollary 9.2 the boundedness of S, (12.14), is equivalent to the existence of the operators $(I - LV'_{(u)})^{-1}$ as uniformly bounded operators for $u \in U$. The conditions of Theorem 12.1 for $p = V'_{(u)}$ are not strong enough to insure this uniform boundedness with the one exception of condition c.

Therefore, we are now going to assume the conditions *a)* through *e)* in the stronger form that p lies in a closed interval for which these conditions hold:

$\bar{a})$ κ_n and κ_{n+1} (κ_{-n} and $\kappa_{-(n+1)}$), $n \geqq 1$, exist and $\kappa_n < \alpha_n \leqq p \leqq \alpha_{n+1} < \kappa_{n+1}$ ($\kappa_{-n} > \alpha_{-n} \geqq p \geqq \alpha_{-(n+1)} > \kappa_{-(n+1)}$)

$\bar{b})$ κ_n (κ_{-n}) exists as the largest positive (smallest negative) characteristic value and $p \geqq \alpha_n > \kappa_n$ ($p \leqq \alpha_{-n} < \kappa_{-n}$)

$\bar{c})$ There is no positive (negative) characteristic value and $p \geqq 0$ ($p \leqq 0$).

$\bar{d})$ κ_1 (κ_{-1}) exists and $0 \leqq p \leqq \alpha_1 < \kappa_1$ ($\kappa_{-1} < \alpha_{-1} \leqq p \leqq 0$).

$\bar{e})$ $\|p\| \leqq \alpha < \min\limits_i (|\kappa_i|)$.

Here κ_i are the characteristic values of A according to (12.6) and α, α_i are real constants.

Then, instead of Theorem 12.1, we have

THEOREM 12.4. Let A be a linear completely continuous symmetric operator on a Hilbert space H into H, let κ_i be its characteristic values (according to (12.5), (12.6)), and let $p \in P$. Finally, let one of the above conditions $\bar{a})$ through $\bar{e})$ be satisfied.

Then the inequality

$$|\mu_i - 1| \geqq m > 0, \tag{12.14}$$

holds for the eigenvalues μ_i of Ap where m is a constant which does not depend on p but only on the interval $[\alpha_i, \alpha_j]$ in which p is assumed to lie according to the conditions $\bar{a})$ ... $\bar{e})$.

The proof is quite similar to the proof[1]) of Theorem 12.1 and may be left to the reader.

From Theorem 12.4 it follows that, under its assumptions, the norm of $I - Ap$ has a positive lower bound. To prove this fact we assume first that $p > 0$, $p \in P$. Then also $\sqrt{p} > 0$, by definition of P, that is, $\sqrt{p}u = \theta$ implies $u = \theta$, or $\sqrt{p}^{-1}$ exists.[2]) Since $\sqrt{p}^{-1}$ has a bounded inverse [3]), namely $\sqrt{p}$,

$$\| \sqrt{p}^{-1} u \| \geqq k \| u \|, \; k > 0, \text{ for all } u \in H. \tag{12.15}$$

[1]) For instance, in the first case $\bar{a})$ we get the inequality $\mu_{n+1} \leqq \sigma_{n+1} < 1 < \sigma_n \leqq \mu_n$ where μ_i, σ_{n+1}, σ_n are the eigenvalues of the operators Ap, $A\alpha_{n+1}$, $A\alpha_n$, respectively.

[2]) $\sqrt{p}^{-1}$ is not necessarily in P.

[3]) E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

Let $\{\Psi_i\}$ be the set of orthonormal eigenvectors of the operator $C = \sqrt{p}A\sqrt{p}$ corresponding to the eigenvalues μ_i of C which are also the eigenvalues of the operator Ap, as already mentioned above. Let u be an arbitrary element in H, $\| u \| = 1$, and $\sqrt{p}u = \sum_i c_i \Psi_i$ where the sum includes the term $c_0 \Psi_0$ in which $C\Psi_0 = \theta$ and $\| \Psi_0 \| = 1$.

Then (12.14) and (12.15) yield

$$\| (I - Ap) u \|^2 = \| \sqrt{p}^{-1} (I - C) \sqrt{pu} \|^2 \geqq k^2 \| (I - C) \sqrt{pu} \|^2$$
$$= k^2 \sum_i | c_i |^2 | 1 - \mu_i |^2 \geqq k^2 \min (m^2, 1) = \tilde{m}^2 > 0 .$$

Hence

$$\| I - Ap \| \geqq \tilde{m} > 0 .$$

If $p \geqq 0$, i.e. $(pu, u) \geqq 0$ for $u \neq \theta$, then each $u \in H$ is either in the null space, N, of $\sqrt{p}$, i.e. $\sqrt{pu} = \theta$, or it is not. We then consider classes of elements by defining u_1, u_2 to belong to the same class $\bar{u}_c$, briefly $u_1 \equiv u_2$, if and only if $u_1 - u_2 \in N$. Then it follows immediately from $u_1 \equiv u_2$ that $\sqrt{pu_1} \equiv \sqrt{pu_2}$, and vice-versa. Since also $(I - Ap) N = N$ we may regard the operator $\sqrt{p}$ as an operator on the Hilbert space spanned by the congruence classes modulo N, represented by one arbitrary element, $\bar{u}$, of each class. In other words we identify the elements of each class. Thus we have $\sqrt{p\bar{u}} = \theta$ implies $u \in N$, i.e. that $\sqrt{p}^{-1}$ exists, and we can repeat our above argument in the case $\bar{u}_c \neq N$, i.e. $\bar{u} \notin N$.

If $u \in N$ we simply have

$$\| (I - Ap) u \| = \| u \| .$$

The cases $p \leqq 0$ can be treated, as above, by considering the operator $\tilde{A}(-p) = -A(-p)$.

Hence, under the assumptions of Theorem 12.4 we have

$$\| I - Ap \| \geqq c > 0 . \qquad (12.16)$$

These considerations together with Theorem 12.3, setting $L = A$ and $pv = p(u)\, v = V'_{(u)}\, v$, yield the

THEOREM 12.5. Let the product operator LV with a linear completely continuous symmetric operator L and a continuously differentiable operator V be defined on a Hilbert space H and have its range in H.

Let κ_i be the characteristic values of $L = A$ according to (12.5) and (12.6), and let $V'_{(u)}\, v = pu$, $p \in P$, satisfy one of the conditions $\bar{a})$ through $\bar{e})$ (as defined for Theorem 12.4) for each $u \in H$.

Then the equation

$$u = LVu + w,$$

has a solution for each $w \in H$.

This theorem generalizes, for example, some existence theorems for non-linear integral equations of the Hammerstein type, that is, equations of the form [1])

$$u(x) + \int_{\mathfrak{L}} K(x, y) f(y, u(y))\, dy = g(x), \qquad (12.17)$$

where x, y are n-dimensional vectors and $\mathfrak{L}$ is a region in R^n; viz., no definiteness of the kernel K is required and the derivative $f_u(x, u)$ need not be bounded by the least characteristic value k_1.

Example. The problem $-y'' = f(x, y)$, $y(a) = A$, $y(b) = B$, $(b > a)$, is solvable if, for instance, the function f is continuous and continuously differentiable with respect to y in the strip $a \leqq x \leqq b$, $|y| < \infty$, and if $f_y(x, y)$ satisfies there one of the conditions: [2])

$$|f_y(x, y)| \leqq \alpha < \frac{\pi^2}{(b-a)^2}; \quad \textit{or } f_y < 0;$$

or

$$\frac{n^2 \pi^2}{(b-a)^2} < \alpha_n \leqq f_y(x, y) \leqq \alpha_{n+1} < \frac{(n+1)^2 \pi^2}{(b-a)^2}.$$

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1) A. Hammerstein [22], see also F. G. Tricomi [23], section 4.6.

2) The known theorems usually cover only the first two cases of this special example. See F. Lettenmeyer [24] and H. Epheser [25]. These papers are more general in another direction.

The proof follows immediately from Theorem 12.5 by writing the problem in the form (12.17). In this case the operator L happens to be definite. But this is not required or used in the proof.

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