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Artikel: ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS

Autor: Ehrmann, Hans H.

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$$
\begin{aligned}
\Delta_2 \Delta_1 T_0 &= \Delta_1 \Delta_2 T_0 \\
&= T(x_0 + h_2 + h_1) - T(x_0 + h_1) - T(x_0 + h_2) - T(x_0) \\
&= T_x(x_0 + h_1)\, h_2 - T_x(x_0)\, h_2 + \circ(h_2) \\
&= T_{xx}\, h_1\, h_2 + \circ(h_1) + \circ(h_2),
\end{aligned}
$$

and

$$\Delta_i\, T'_{0(u_0)} = T_u(x_0 + h_i, u_0) - T_u(x_0, u_0) = T_{xu}\, h_i + \circ(h_i), \quad i = 1, 2.$$

Hence by (8.9) and (8.10), neglecting the terms $\circ(h_i)$, it results

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\begin{aligned}
\varphi''(x_0)\, h_2\, h_1 = &-(T_u)^{-1} \{ T_{xu}(h_1 [\varphi'(x_0)\, h_2] + h_2 [\varphi'(x_0)\, h_1]) \\
&+ T_{xx}\, h_1\, h_2 + T_{uu} [\varphi'(x_0)\, h_1][\varphi'(x_0)\, h_2] \}
\end{aligned}
$$

where the derivatives of T are taken at the point (x_0, u_0) [e.g. $T_u = T_u(x_0, u_0)$] and, for example, $T_{xu}\, hk$ means that the bilinear operator $T_{xu} = T_{xu}(x_0, u_0)$ applies to the elements h and k. Here $\varphi'(x_0)\, h$ can be expressed by $-T_u^{-1}\, T_x\, h$ according to (8.7).

9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR

In this chapter a method for the proof of the existence of a solution of a non-linear equation

$$Tu = \theta, \tag{9.1}$$

is introduced which may be useful in cases where T has a derivative but cannot be written in the form I-V with completely continuous operator V or in which the complete continuity of V is difficult to show.

THEOREM 9.1. Assume T is a closed[1]) operator defined on an (open) domain $D \subset B_1$ and there has a derivative $T'_{(u)}$ such that $T'_{(u)} - T'_{(v)}$ $(u, v \in D)$ is bounded and continuous[2]) with respect to u. The range of T lies in B_2.

1) See, for example, E. Hille and R. S. Phillips [4], p. 40, or N.I. Achieser and I. M. Glasmann [14], p. 82.

2) We don't require that $T'_{(u)}k$ is continuous with respect to k.

Let T_0 be any operator on $D_0 \supset D$ into B_2 with the properties:

a. $$T_0 u_0 = \theta \quad \text{for some} \quad u_0 \in D\,. \tag{9.2}$$

b. T_0 has a derivative $T'_{0(u)}$ in D satisfying the same conditions as $T'_{(u)}$

c. The operators
$$T_\lambda = (1-\lambda)\, T_0 + \lambda T, \quad 0 < \lambda < 1\,,$$
are closed.

Denote
$$U = \{u \colon T_\lambda u = \theta\,,\ 0 \leqq \lambda < 1\}\,.$$

Then either (9.1) has a solution or [1]) the sets

$$S = \{s \colon s = \frac{\| k \|}{\| T'_{\lambda\,(u)}\, k \|}\,,\ k \in B_1\,,\ u \in U\,,\ 0 \leqq \lambda < 1\}\,, \tag{9.3}$$

and

$$V = \{v \colon v = \| (T - T_0)\, u \|\,, \quad u \in U\}\,, \tag{9.4}$$

are not both bounded.

Proof. Let Λ be the set of all λ in $0 \leqq \lambda \leqq 1$ for which the equation $T_\lambda u = \theta$ has a solution. Then $\Lambda \neq \emptyset$ because $0 \in \Lambda$. Let S be bounded:

$$s \leqq C_1 \quad \text{or} \quad \| T'_{\lambda\,(u)}\, k \| \geqq \frac{1}{C_1} \| k \|\,, \quad \frac{1}{C_1} > 0\,.$$

Therefore [2]), the operator $T'_{\lambda(u)}$ has a bounded inverse $T'^{-1}_{\lambda(u)}$ and

$$\| T'^{-1}_{\lambda(u)} \| \leqq C_1\,. \tag{9.5}$$

Hence the assumptions of Theorem 7.1, supplement 7.1 *b*, are satisfied. Therefore, it follows that the set Λ is open with respect to [0, 1].

Moreover, Theorem 8.1 says that each " point " $(T_\lambda,\ u(T_\lambda))$, $u \in U$, has an Ω-neighborhood in which $u = u(T)$ is unique, continuous and differentiable if assumption *A* of Chapter *8* is satisfied. This is obviously true if we restrict ourselves to

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1) The statements shall not exclude each other, i.e. at least one of them is true.
2) See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

$T_\lambda \in \Omega$. Then the operator $\tilde{T}'_{(u)}$ in (8.3) becomes $T'_{\lambda(u)}$. From this it follows that we can construct a unique and continuously differentiable function $\varphi(\lambda) = u(T_\lambda) \in D$ with $T_\lambda \varphi(\lambda) = \theta$ defined on some interval $0 \leqq \lambda < \tilde{\lambda}$ if we apply the Theorems 7.1 and 8.1 repeatedly. Let $[0, \tilde{\lambda}]$ be the largest interval for which $\varphi(\lambda)$ can be defined by this construction under the assumption that (9.1) is not solvable, i.e. $1 \notin \Lambda$. Obviously $0 < \tilde{\lambda} < 1$ and $\tilde{\lambda} \notin \Lambda$.

Then by (8.7) we have

$$\varphi'(\lambda) = -T'^{-1}_{\lambda(\varphi(\lambda))} T'_{(\lambda)(\varphi(\lambda))} = -T'^{-1}_{\lambda(\varphi(\lambda))}(T - T_0)\, u(T_\lambda) \tag{9.6}$$

for $0 \leqq \lambda < \tilde{\lambda}$. And $\varphi'(\lambda)$ is a bounded linear operator on R^1 into B_1.

Now let $\lambda_\nu < \tilde{\lambda}$, $\nu = 1, 2, \ldots$, be a sequence converging to $\tilde{\lambda}$ and $u_\nu = u(T_{\lambda_\nu}) = \varphi(\lambda_\nu)$ be the solutions of $T_{\lambda_\nu} u = \theta$ as just obtained. Then by the mean value theorem of the differential calculus we have, for $\lambda_\mu > \lambda_\nu$,

$$\| u_\nu - u_\mu \| \leqq \sup_{\lambda_\nu \leqq \lambda \leqq \lambda_\mu} \| \varphi'(\lambda) \| \, | \lambda_\mu - \lambda_\nu | .$$

If we assume that the sets S and V in (9.3), (9.4), respectively, are bounded with bounds C_1 and C_2 then by (9.5) and (9.6)

$$\| u_\nu - u_\mu \| \leqq C_1 C_2 \, | \lambda_\mu - \lambda_\nu | , \qquad \mu, \nu = 1, 2, \ldots .$$

Hence $\{ u_\nu \}$ is a Cauchy sequence and by the completeness of B_1 there exists a limit element $\tilde{u} \in B_1$:

$$\tilde{u} = \lim_{\nu \to \infty} u_\nu .$$

Because $u_\nu \in D$ and $T_{\lambda_\nu} u_\nu = \theta$, $\nu = 1, 2, \ldots$, we have

$$\| T_{\tilde{\lambda}} u_\nu \| = \| (T_{\tilde{\lambda}} - T_{\lambda_\nu}) u_\nu \| = \| (\tilde{\lambda} - \lambda_\nu)(T - T_0) u_\nu \|$$
$$\leqq | \tilde{\lambda} - \lambda_\nu | \, \| (T - T_0) u_\nu \| .$$

By (9.4) and $\lambda_\nu \to \tilde{\lambda}$, $\nu \to \infty$, we have

$$\| T_{\tilde{\lambda}} u_\nu \| \to 0 \quad \text{for} \quad u_\nu \in D, \; u_\nu \to \tilde{u} .$$

Since $T_{\tilde{\lambda}}$ is closed, then

$$\tilde{u} \in D \quad \text{and} \quad T_{\tilde{\lambda}}\, \tilde{u} = \theta .$$

Therefore, Λ also is closed with respect to [0, 1]. Thus $\Lambda = [0.\ 1]$ which completes the proof.

If we choose, in particular, $T_0\, u = Tu - Tu_0$ for some fixed $u_0 \in D$, we get

$$T_\lambda u = Tu - (1-\lambda)\, Tu_0 \quad \text{and} \quad T - T_0 = Tu_0 = \text{const.} \tag{9.7}$$

Thus, all assumptions on T_0 and also the boundedness of the set V are satisfied automatically, and we have the

Corollary 9.1. Assume T is a closed operator defined on an (open) domain $D \subset B_1$ and with range in B_2. Let T have a derivative $T'_{(u)}$ there such that $T'_{(u)} - T'_{(v)}$ is a bounded operator depending continuously on u, $(u,\ v \in D)$.

Then either (9.1) has a solution or the set S in (9.3) is not bounded.

The condition of the boundedness of the set S is equivalent to the condition

$$\inf \{ \| T'_{\lambda\,(u)}\, k \| : \| k \| = 1 ,\quad k \in B_1 ,\quad u \in U ,\quad 0 \leqq \lambda < 1 \} = m > 0 . \tag{9.8}$$

Since $\lambda = 0$ is not excluded there is no statement if $T'_{0(u)}\, k$ is θ for some k; for example, if T_0 is constant. As (9.8) or the boundedness of S is equivalent[1]) also to the existence of a bounded inverse of $T'_{\lambda(u)}$ the existence of a solution of (9.1) can only fail if $T'^{-1}_{\lambda(u)}$ fails to exist as a bounded operator for some $\lambda \in [0,\ 1]$. The proof of Theorem 9.1 shows that we even can restrict ourselves to examine only $T'^{-1}_{\lambda(u)}$ for $u = \varphi(\lambda)$ or according to formula (8.6) to $(T_\lambda^{-1})'_{(\theta)} = (T'_{\lambda(\varphi(\lambda))})^{-1}$. Thus, writing (9.1) in the form

$$Tu = w_1 , \tag{9.9}$$

and choosing $T_0\, u = Tu - w_0$, $w_0 = Tu_0$, as for (9.7), we get $T_\lambda\, u = Tu - w_0 - \lambda\,(w_1 - w_0)$ and we have the

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1) See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

Corollary 9.2. The equation (9.9) with T satisfying the assumptions of Theorem 9.1 has at least one solution if for at least one $u_0 \in D$, with $\varphi(\lambda)$ the same as in the proof of Theorem 9.1, and

$$w(\lambda) = w_0 + \lambda(w_1 - w_0), \tag{9.10}$$

the operators

$$(T'_{(\varphi(\lambda))})^{-1} = (T^{-1})'_{(w(\lambda))}, \quad 0 \leqq \lambda < 1,$$

exist and are bounded uniformly in λ, or equivalently, if $T'^{-1}_{(u_0)}$ exists as a bounded operator and

$$\| (T'_{(\varphi(\lambda))})^{-1} \| = \| (T^{-1})'_{(w(\lambda))} \|,$$

remains finite with increasing λ from 0 to 1.

Example. It is well known that the equation

$$Tz \equiv \tan z = w, \ z, \ w \text{ complex numbers},$$

is not solvable only for $w = \pm i$. Theorem 9.1 immediately shows that the equation is solvable for all $w \neq \pm i$. For

$$(T^{-1})'_{(w)} = \frac{1}{1+w^2},$$

and, with $w_{0_1} = 0 = \tan 0$ and $w_{0_2} = 1 = \tan \frac{\pi}{4}$, all points of the complex number plane can be reached on straight lines (9.10) from either 0 or 1 such that $\frac{1}{1+(w(\lambda))^2}$ remains bounded with the only exceptions $w = \pm i$.

10. Completely continuous operators, neighborhood and inverse function theorems.

The assumptions of the theorems can be partially weakened if the non-linear equation can be written in the form

$$u = Vu, \tag{10.1}$$

with a completely continuous operator V. Complete continuity