

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 9 (1963)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS  
**Autor:** Ehrmann, Hans H.  
**Kapitel:** 9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR  
**DOI:** <https://doi.org/10.5169/seals-38780>

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$$\begin{aligned}
 \Delta_2 \Delta_1 T_0 &= \Delta_1 \Delta_2 T_0 \\
 &= T(x_0 + h_2 + h_1) - T(x_0 + h_1) - T(x_0 + h_2) + T(x_0) \\
 &= T_x(x_0 + h_1) h_2 - T_x(x_0) h_2 + o(h_2) \\
 &= T_{xx} h_1 h_2 + o(h_1) + o(h_2),
 \end{aligned}$$

and

$$\Delta_i T'_{0(u_0)} = T_u(x_0 + h_i, u_0) - T_u(x_0, u_0) = T_{xu} h_i + o(h_i), \quad i = 1, 2.$$

Hence by (8.9) and (8.10), neglecting the terms  $o(h_i)$ , it results

$$\begin{aligned}
 \varphi''(x_0) h_2 h_1 &= -(T_u)^{-1} \{ T_{xu} (h_1 [\varphi'(x_0) h_2] + h_2 [\varphi'(x_0) h_1]) \\
 &\quad + T_{xx} h_1 h_2 + T_{uu} [\varphi'(x_0) h_1] [\varphi'(x_0) h_2] \}
 \end{aligned}$$

where the derivatives of  $T$  are taken at the point  $(x_0, u_0)$  [e.g.  $T_u = T_u(x_0, u_0)$ ] and, for example,  $T_{xu} h k$  means that the bilinear operator  $T_{xu} = T_{xu}(x_0, u_0)$  applies to the elements  $h$  and  $k$ . Here  $\varphi'(x_0) h$  can be expressed by  $-T_u^{-1} T_x h$  according to (8.7).

## 9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR

In this chapter a method for the proof of the existence of a solution of a non-linear equation

$$Tu = \theta, \quad (9.1)$$

is introduced which may be useful in cases where  $T$  has a derivative but cannot be written in the form  $I - V$  with completely continuous operator  $V$  or in which the complete continuity of  $V$  is difficult to show.

**THEOREM 9.1.** Assume  $T$  is a closed<sup>1)</sup> operator defined on an (open) domain  $D \subset B_1$  and there has a derivative  $T'_{(u)}$  such that  $T'_{(u)} - T'_{(v)}$  ( $u, v \in D$ ) is bounded and continuous<sup>2)</sup> with respect to  $u$ . The range of  $T$  lies in  $B_2$ .

<sup>1)</sup> See, for example, E. Hille and R. S. Phillips [4], p. 40, or N. I. Achieser and I. M. Glasmann [14], p. 82.

<sup>2)</sup> We don't require that  $T'_{(u)}k$  is continuous with respect to  $k$ .

Let  $T_0$  be any operator on  $D_0 \supset D$  into  $B_2$  with the properties:

$$a. \quad T_0 u_0 = \theta \quad \text{for some } u_0 \in D. \quad (9.2)$$

b.  $T_0$  has a derivative  $T'_{0(u)}$  in  $D$  satisfying the same conditions as  $T'_{(u)}$

c. The operators

$$T_\lambda = (1 - \lambda) T_0 + \lambda T, \quad 0 < \lambda < 1,$$

are closed.

Denote

$$U = \{u: T_\lambda u = \theta, \quad 0 \leq \lambda < 1\}.$$

Then either (9.1) has a solution or<sup>1)</sup> the sets

$$S = \{s: s = \frac{\|k\|}{\|T'_{\lambda(u)} k\|}, \quad k \in B_1, \quad u \in U, \quad 0 \leq \lambda < 1\}, \quad (9.3)$$

and

$$V = \{v: v = \|(T - T_0)u\|, \quad u \in U\}, \quad (9.4)$$

are not both bounded.

*Proof.* Let  $A$  be the set of all  $\lambda$  in  $0 \leq \lambda \leq 1$  for which the equation  $T_\lambda u = \theta$  has a solution. Then  $A \neq \emptyset$  because  $0 \in A$ . Let  $S$  be bounded:

$$s \leq C_1 \quad \text{or} \quad \|T'_{\lambda(u)} k\| \geq \frac{1}{C_1} \|k\|, \quad \frac{1}{C_1} > 0.$$

Therefore<sup>2)</sup>, the operator  $T'_{\lambda(u)}$  has a bounded inverse  $T'^{-1}_{\lambda(u)}$  and

$$\|T'^{-1}_{\lambda(u)}\| \leq C_1. \quad (9.5)$$

Hence the assumptions of Theorem 7.1, supplement 7.1 b, are satisfied. Therefore, it follows that the set  $A$  is open with respect to  $[0, 1]$ .

Moreover, Theorem 8.1 says that each "point"  $(T_\lambda, u(T_\lambda))$ ,  $u \in U$ , has an  $\Omega$ -neighborhood in which  $u = u(T)$  is unique, continuous and differentiable if assumption A of Chapter 8 is satisfied. This is obviously true if we restrict ourselves to

<sup>1)</sup> The statements shall not exclude each other, i.e. at least one of them is true.

<sup>2)</sup> See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

$T_\lambda \in \Omega$ . Then the operator  $\tilde{T}'_{(u)}$  in (8.3) becomes  $T'_{\lambda(u)}$ . From this it follows that we can construct a unique and continuously differentiable function  $\varphi(\lambda) = u(T_\lambda) \in D$  with  $T_\lambda \varphi(\lambda) = \theta$  defined on some interval  $0 \leq \lambda < \tilde{\lambda}$  if we apply the Theorems 7.1 and 8.1 repeatedly. Let  $[0, \tilde{\lambda}]$  be the largest interval for which  $\varphi(\lambda)$  can be defined by this construction under the assumption that (9.1) is not solvable, i.e.  $1 \notin A$ . Obviously  $0 < \tilde{\lambda} < 1$  and  $\tilde{\lambda} \notin A$ .

Then by (8.7) we have

$$\varphi'(\lambda) = -T_{\lambda(\varphi(\lambda))}^{-1} T'_{(\lambda)(\varphi(\lambda))} = -T_{\lambda(\varphi(\lambda))}^{-1} (T - T_0) u(T_\lambda) \quad (9.6)$$

for  $0 \leq \lambda < \tilde{\lambda}$ . And  $\varphi'(\lambda)$  is a bounded linear operator on  $R^1$  into  $B_1$ .

Now let  $\lambda_\nu < \tilde{\lambda}$ ,  $\nu = 1, 2, \dots$ , be a sequence converging to  $\tilde{\lambda}$  and  $u_\nu = u(T_{\lambda_\nu}) = \varphi(\lambda_\nu)$  be the solutions of  $T_{\lambda_\nu} u = \theta$  as just obtained. Then by the mean value theorem of the differential calculus we have, for  $\lambda_\mu > \lambda_\nu$ ,

$$\|u_\nu - u_\mu\| \leq \sup_{\lambda_\nu \leq \lambda \leq \lambda_\mu} \|\varphi'(\lambda)\| |\lambda_\mu - \lambda_\nu|.$$

If we assume that the sets  $S$  and  $V$  in (9.3), (9.4), respectively, are bounded with bounds  $C_1$  and  $C_2$  then by (9.5) and (9.6)

$$\|u_\nu - u_\mu\| \leq C_1 C_2 |\lambda_\mu - \lambda_\nu|, \quad \mu, \nu = 1, 2, \dots$$

Hence  $\{u_\nu\}$  is a Cauchy sequence and by the completeness of  $B_1$  there exists a limit element  $\tilde{u} \in B_1$ :

$$\tilde{u} = \lim_{\nu \rightarrow \infty} u_\nu.$$

Because  $u_\nu \in D$  and  $T_{\lambda_\nu} u_\nu = \theta$ ,  $\nu = 1, 2, \dots$ , we have

$$\begin{aligned} \|T_{\tilde{\lambda}} u_\nu\| &= \|(T_{\tilde{\lambda}} - T_{\lambda_\nu}) u_\nu\| = \|(\tilde{\lambda} - \lambda_\nu)(T - T_0) u_\nu\| \\ &\leq |\tilde{\lambda} - \lambda_\nu| \|(T - T_0) u_\nu\|. \end{aligned}$$

By (9.4) and  $\lambda_\nu \rightarrow \tilde{\lambda}$ ,  $\nu \rightarrow \infty$ , we have

$$\|T_{\tilde{\lambda}} u_\nu\| \rightarrow 0 \quad \text{for } u_\nu \in D, u_\nu \rightarrow \tilde{u}.$$

Since  $T_{\tilde{\lambda}}$  is closed, then

$$\tilde{u} \in D \quad \text{and} \quad T_{\tilde{\lambda}} \tilde{u} = \theta.$$

Therefore,  $A$  also is closed with respect to  $[0, 1]$ . Thus  $A = [0, 1]$  which completes the proof.

If we choose, in particular,  $T_0 u = Tu - Tu_0$  for some fixed  $u_0 \in D$ , we get

$$T_{\lambda} u = Tu - (1 - \lambda) Tu_0 \quad \text{and} \quad T - T_0 = Tu_0 = \text{const.} \quad (9.7)$$

Thus, all assumptions on  $T_0$  and also the boundedness of the set  $V$  are satisfied automatically, and we have the

*Corollary 9.1.* Assume  $T$  is a closed operator defined on an (open) domain  $D \subset B_1$  and with range in  $B_2$ . Let  $T$  have a derivative  $T'_{(u)}$  there such that  $T'_{(u)} - T'_{(v)}$  is a bounded operator depending continuously on  $u, (u, v \in D)$ .

Then either (9.1) has a solution or the set  $S$  in (9.3) is not bounded.

The condition of the boundedness of the set  $S$  is equivalent to the condition

$$\inf \{ \| T'_{\lambda(u)} k \| : \| k \| = 1, \quad k \in B_1, \quad u \in U, \quad 0 \leq \lambda < 1 \} \\ = m > 0. \quad (9.8)$$

Since  $\lambda = 0$  is not excluded there is no statement if  $T'_{0(u)} k$  is  $\theta$  for some  $k$ ; for example, if  $T_0$  is constant. As (9.8) or the boundedness of  $S$  is equivalent<sup>1)</sup> also to the existence of a bounded inverse of  $T'_{\lambda(u)}$  the existence of a solution of (9.1) can only fail if  $T'_{\lambda(u)}$  fails to exist as a bounded operator for some  $\lambda \in [0, 1]$ . The proof of Theorem 9.1 shows that we even can restrict ourselves to examine only  $T'^{-1}_{\lambda(u)}$  for  $u = \varphi(\lambda)$  or according to formula (8.6) to  $(T'^{-1}_{\lambda})_{(\theta)} = (T'_{\lambda(\varphi(\lambda))})^{-1}$ . Thus, writing (9.1) in the form

$$Tu = w_1, \quad (9.9)$$

and choosing  $T_0 u = Tu - w_0$ ,  $w_0 = Tu_0$ , as for (9.7), we get  $T_{\lambda} u = Tu - w_0 - \lambda(w_1 - w_0)$  and we have the

<sup>1)</sup> See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

*Corollary 9.2.* The equation (9.9) with  $T$  satisfying the assumptions of Theorem 9.1 has at least one solution if for at least one  $u_0 \in D$ , with  $\varphi(\lambda)$  the same as in the proof of Theorem 9.1, and

$$w(\lambda) = w_0 + \lambda(w_1 - w_0), \quad (9.10)$$

the operators

$$(T'_{(\varphi(\lambda))})^{-1} = (T^{-1})'_{(w(\lambda))}, \quad 0 \leq \lambda < 1,$$

exist and are bounded uniformly in  $\lambda$ , or equivalently, if  $T'^{-1}_{(u_0)}$  exists as a bounded operator and

$$\|(T'_{(\varphi(\lambda))})^{-1}\| = \|(T^{-1})'_{(w(\lambda))}\|,$$

remains finite with increasing  $\lambda$  from 0 to 1.

*Example.* It is well known that the equation

$$Tz \equiv \tan z = \omega, \quad z, \omega \text{ complex numbers,}$$

is not solvable only for  $\omega = \pm i$ . Theorem 9.1 immediately shows that the equation is solvable for all  $\omega \neq \pm i$ . For

$$(T^{-1})'_{(w)} = \frac{1}{1 + w^2},$$

and, with  $\omega_{0_1} = 0 = \tan 0$  and  $\omega_{0_2} = 1 = \tan \frac{\pi}{4}$ , all points of the complex number plane can be reached on straight lines (9.10) from either 0 or 1 such that  $\frac{1}{1 + (\omega(\lambda))^2}$  remains bounded with the only exceptions  $\omega = \pm i$ .

## 10. COMPLETELY CONTINUOUS OPERATORS, NEIGHBORHOOD AND INVERSE FUNCTION THEOREMS.

The assumptions of the theorems can be partially weakened if the non-linear equation can be written in the form

$$u = Vu, \quad (10.1)$$

with a completely continuous operator  $V$ . Complete continuity