

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 9 (1963)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS
Autor: Ehrmann, Hans H.
Kapitel: 9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR
DOI: <https://doi.org/10.5169/seals-38780>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 24.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$$\begin{aligned}
 \Delta_2 \Delta_1 T_0 &= \Delta_1 \Delta_2 T_0 \\
 &= T(x_0 + h_2 + h_1) - T(x_0 + h_1) - T(x_0 + h_2) + T(x_0) \\
 &= T_x(x_0 + h_1) h_2 - T_x(x_0) h_2 + o(h_2) \\
 &= T_{xx} h_1 h_2 + o(h_1) + o(h_2),
 \end{aligned}$$

and

$$\Delta_i T'_{0(u_0)} = T_u(x_0 + h_i, u_0) - T_u(x_0, u_0) = T_{xu} h_i + o(h_i), \quad i = 1, 2.$$

Hence by (8.9) and (8.10), neglecting the terms $o(h_i)$, it results

$$\begin{aligned}
 \varphi''(x_0) h_2 h_1 &= -(T_u)^{-1} \{ T_{xu} (h_1 [\varphi'(x_0) h_2] + h_2 [\varphi'(x_0) h_1]) \\
 &\quad + T_{xx} h_1 h_2 + T_{uu} [\varphi'(x_0) h_1] [\varphi'(x_0) h_2] \}
 \end{aligned}$$

where the derivatives of T are taken at the point (x_0, u_0) [e.g. $T_u = T_u(x_0, u_0)$] and, for example, $T_{xu} h k$ means that the bilinear operator $T_{xu} = T_{xu}(x_0, u_0)$ applies to the elements h and k . Here $\varphi'(x_0) h$ can be expressed by $-T_u^{-1} T_x h$ according to (8.7).

9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR

In this chapter a method for the proof of the existence of a solution of a non-linear equation

$$Tu = \theta, \quad (9.1)$$

is introduced which may be useful in cases where T has a derivative but cannot be written in the form $I - V$ with completely continuous operator V or in which the complete continuity of V is difficult to show.

THEOREM 9.1. Assume T is a closed¹⁾ operator defined on an (open) domain $D \subset B_1$ and there has a derivative $T'_{(u)}$ such that $T'_{(u)} - T'_{(v)}$ ($u, v \in D$) is bounded and continuous²⁾ with respect to u . The range of T lies in B_2 .

¹⁾ See, for example, E. Hille and R. S. Phillips [4], p. 40, or N. I. Achieser and I. M. Glasmann [14], p. 82.

²⁾ We don't require that $T'_{(u)}k$ is continuous with respect to k .

Let T_0 be any operator on $D_0 \supset D$ into B_2 with the properties:

$$a. \quad T_0 u_0 = \theta \quad \text{for some } u_0 \in D. \quad (9.2)$$

b. T_0 has a derivative $T'_{0(u)}$ in D satisfying the same conditions as $T'_{(u)}$

c. The operators

$$T_\lambda = (1 - \lambda) T_0 + \lambda T, \quad 0 < \lambda < 1,$$

are closed.

Denote

$$U = \{u: T_\lambda u = \theta, \quad 0 \leq \lambda < 1\}.$$

Then either (9.1) has a solution or¹⁾ the sets

$$S = \{s: s = \frac{\|k\|}{\|T'_{\lambda(u)} k\|}, \quad k \in B_1, \quad u \in U, \quad 0 \leq \lambda < 1\}, \quad (9.3)$$

and

$$V = \{v: v = \|(T - T_0)u\|, \quad u \in U\}, \quad (9.4)$$

are not both bounded.

Proof. Let A be the set of all λ in $0 \leq \lambda \leq 1$ for which the equation $T_\lambda u = \theta$ has a solution. Then $A \neq \emptyset$ because $0 \in A$. Let S be bounded:

$$s \leq C_1 \quad \text{or} \quad \|T'_{\lambda(u)} k\| \geq \frac{1}{C_1} \|k\|, \quad \frac{1}{C_1} > 0.$$

Therefore²⁾, the operator $T'_{\lambda(u)}$ has a bounded inverse $T'^{-1}_{\lambda(u)}$ and

$$\|T'^{-1}_{\lambda(u)}\| \leq C_1. \quad (9.5)$$

Hence the assumptions of Theorem 7.1, supplement 7.1 b, are satisfied. Therefore, it follows that the set A is open with respect to $[0, 1]$.

Moreover, Theorem 8.1 says that each "point" $(T_\lambda, u(T_\lambda))$, $u \in U$, has an Ω -neighborhood in which $u = u(T)$ is unique, continuous and differentiable if assumption A of Chapter 8 is satisfied. This is obviously true if we restrict ourselves to

¹⁾ The statements shall not exclude each other, i.e. at least one of them is true.

²⁾ See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

$T_\lambda \in \Omega$. Then the operator $\tilde{T}'_{(u)}$ in (8.3) becomes $T'_{\lambda(u)}$. From this it follows that we can construct a unique and continuously differentiable function $\varphi(\lambda) = u(T_\lambda) \in D$ with $T_\lambda \varphi(\lambda) = \theta$ defined on some interval $0 \leq \lambda < \tilde{\lambda}$ if we apply the Theorems 7.1 and 8.1 repeatedly. Let $[0, \tilde{\lambda}]$ be the largest interval for which $\varphi(\lambda)$ can be defined by this construction under the assumption that (9.1) is not solvable, i.e. $1 \notin A$. Obviously $0 < \tilde{\lambda} < 1$ and $\tilde{\lambda} \notin A$.

Then by (8.7) we have

$$\varphi'(\lambda) = -T_{\lambda(\varphi(\lambda))}^{-1} T'_{(\lambda)(\varphi(\lambda))} = -T_{\lambda(\varphi(\lambda))}^{-1} (T - T_0) u(T_\lambda) \quad (9.6)$$

for $0 \leq \lambda < \tilde{\lambda}$. And $\varphi'(\lambda)$ is a bounded linear operator on R^1 into B_1 .

Now let $\lambda_\nu < \tilde{\lambda}$, $\nu = 1, 2, \dots$, be a sequence converging to $\tilde{\lambda}$ and $u_\nu = u(T_{\lambda_\nu}) = \varphi(\lambda_\nu)$ be the solutions of $T_{\lambda_\nu} u = \theta$ as just obtained. Then by the mean value theorem of the differential calculus we have, for $\lambda_\mu > \lambda_\nu$,

$$\|u_\nu - u_\mu\| \leq \sup_{\lambda_\nu \leq \lambda \leq \lambda_\mu} \|\varphi'(\lambda)\| |\lambda_\mu - \lambda_\nu|.$$

If we assume that the sets S and V in (9.3), (9.4), respectively, are bounded with bounds C_1 and C_2 then by (9.5) and (9.6)

$$\|u_\nu - u_\mu\| \leq C_1 C_2 |\lambda_\mu - \lambda_\nu|, \quad \mu, \nu = 1, 2, \dots$$

Hence $\{u_\nu\}$ is a Cauchy sequence and by the completeness of B_1 there exists a limit element $\tilde{u} \in B_1$:

$$\tilde{u} = \lim_{\nu \rightarrow \infty} u_\nu.$$

Because $u_\nu \in D$ and $T_{\lambda_\nu} u_\nu = \theta$, $\nu = 1, 2, \dots$, we have

$$\begin{aligned} \|T_{\tilde{\lambda}} u_\nu\| &= \|(T_{\tilde{\lambda}} - T_{\lambda_\nu}) u_\nu\| = \|(\tilde{\lambda} - \lambda_\nu)(T - T_0) u_\nu\| \\ &\leq |\tilde{\lambda} - \lambda_\nu| \|(T - T_0) u_\nu\|. \end{aligned}$$

By (9.4) and $\lambda_\nu \rightarrow \tilde{\lambda}$, $\nu \rightarrow \infty$, we have

$$\|T_{\tilde{\lambda}} u_\nu\| \rightarrow 0 \quad \text{for } u_\nu \in D, u_\nu \rightarrow \tilde{u}.$$

Since $T_{\tilde{\lambda}}$ is closed, then

$$\tilde{u} \in D \quad \text{and} \quad T_{\tilde{\lambda}} \tilde{u} = \theta.$$

Therefore, A also is closed with respect to $[0, 1]$. Thus $A = [0, 1]$ which completes the proof.

If we choose, in particular, $T_0 u = Tu - Tu_0$ for some fixed $u_0 \in D$, we get

$$T_{\lambda} u = Tu - (1 - \lambda) Tu_0 \quad \text{and} \quad T - T_0 = Tu_0 = \text{const.} \quad (9.7)$$

Thus, all assumptions on T_0 and also the boundedness of the set V are satisfied automatically, and we have the

Corollary 9.1. Assume T is a closed operator defined on an (open) domain $D \subset B_1$ and with range in B_2 . Let T have a derivative $T'_{(u)}$ there such that $T'_{(u)} - T'_{(v)}$ is a bounded operator depending continuously on $u, (u, v \in D)$.

Then either (9.1) has a solution or the set S in (9.3) is not bounded.

The condition of the boundedness of the set S is equivalent to the condition

$$\inf \{ \| T'_{\lambda(u)} k \| : \| k \| = 1, \quad k \in B_1, \quad u \in U, \quad 0 \leq \lambda < 1 \} \\ = m > 0. \quad (9.8)$$

Since $\lambda = 0$ is not excluded there is no statement if $T'_{0(u)} k$ is θ for some k ; for example, if T_0 is constant. As (9.8) or the boundedness of S is equivalent¹⁾ also to the existence of a bounded inverse of $T'_{\lambda(u)}$ the existence of a solution of (9.1) can only fail if $T'_{\lambda(u)}$ fails to exist as a bounded operator for some $\lambda \in [0, 1]$. The proof of Theorem 9.1 shows that we even can restrict ourselves to examine only $T'^{-1}_{\lambda(u)}$ for $u = \varphi(\lambda)$ or according to formula (8.6) to $(T'^{-1}_{\lambda})_{(\theta)} = (T'_{\lambda(\varphi(\lambda))})^{-1}$. Thus, writing (9.1) in the form

$$Tu = w_1, \quad (9.9)$$

and choosing $T_0 u = Tu - w_0$, $w_0 = Tu_0$, as for (9.7), we get $T_{\lambda} u = Tu - w_0 - \lambda(w_1 - w_0)$ and we have the

¹⁾ See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

Corollary 9.2. The equation (9.9) with T satisfying the assumptions of Theorem 9.1 has at least one solution if for at least one $u_0 \in D$, with $\varphi(\lambda)$ the same as in the proof of Theorem 9.1, and

$$w(\lambda) = w_0 + \lambda(w_1 - w_0), \quad (9.10)$$

the operators

$$(T'_{(\varphi(\lambda))})^{-1} = (T^{-1})'_{(w(\lambda))}, \quad 0 \leq \lambda < 1,$$

exist and are bounded uniformly in λ , or equivalently, if $T'^{-1}_{(u_0)}$ exists as a bounded operator and

$$\|(T'_{(\varphi(\lambda))})^{-1}\| = \|(T^{-1})'_{(w(\lambda))}\|,$$

remains finite with increasing λ from 0 to 1.

Example. It is well known that the equation

$$Tz \equiv \tan z = \omega, \quad z, \omega \text{ complex numbers,}$$

is not solvable only for $\omega = \pm i$. Theorem 9.1 immediately shows that the equation is solvable for all $\omega \neq \pm i$. For

$$(T^{-1})'_{(w)} = \frac{1}{1 + w^2},$$

and, with $\omega_{0_1} = 0 = \tan 0$ and $\omega_{0_2} = 1 = \tan \frac{\pi}{4}$, all points of the complex number plane can be reached on straight lines (9.10) from either 0 or 1 such that $\frac{1}{1 + (\omega(\lambda))^2}$ remains bounded with the only exceptions $\omega = \pm i$.

10. COMPLETELY CONTINUOUS OPERATORS, NEIGHBORHOOD AND INVERSE FUNCTION THEOREMS.

The assumptions of the theorems can be partially weakened if the non-linear equation can be written in the form

$$u = Vu, \quad (10.1)$$

with a completely continuous operator V . Complete continuity