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$$\begin{aligned}
 \Delta_2 \Delta_1 T_0 &= \Delta_1 \Delta_2 T_0 \\
 &= T(x_0 + h_2 + h_1) - T(x_0 + h_1) - T(x_0 + h_2) + T(x_0) \\
 &= T_x(x_0 + h_1) h_2 - T_x(x_0) h_2 + o(h_2) \\
 &= T_{xx} h_1 h_2 + o(h_1) + o(h_2),
 \end{aligned}$$

and

$$\Delta_i T'_{0(u_0)} = T_u(x_0 + h_i, u_0) - T_u(x_0, u_0) = T_{xu} h_i + o(h_i), \quad i = 1, 2.$$

Hence by (8.9) and (8.10), neglecting the terms $o(h_i)$, it results

$$\begin{aligned}
 \varphi''(x_0) h_2 h_1 &= -(T_u)^{-1} \{ T_{xu} (h_1 [\varphi'(x_0) h_2] + h_2 [\varphi'(x_0) h_1]) \\
 &\quad + T_{xx} h_1 h_2 + T_{uu} [\varphi'(x_0) h_1] [\varphi'(x_0) h_2] \}
 \end{aligned}$$

where the derivatives of T are taken at the point (x_0, u_0) [e.g. $T_u = T_u(x_0, u_0)$] and, for example, $T_{xu} h k$ means that the bilinear operator $T_{xu} = T_{xu}(x_0, u_0)$ applies to the elements h and k . Here $\varphi'(x_0) h$ can be expressed by $-T_u^{-1} T_x h$ according to (8.7).

9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR

In this chapter a method for the proof of the existence of a solution of a non-linear equation

$$Tu = \theta, \tag{9.1}$$

is introduced which may be useful in cases where T has a derivative but cannot be written in the form $I - V$ with completely continuous operator V or in which the complete continuity of V is difficult to show.

THEOREM 9.1. Assume T is a closed¹⁾ operator defined on an (open) domain $D \subset B_1$ and there has a derivative $T'_{(u)}$ such that $T'_{(u)} - T'_{(v)}$ ($u, v \in D$) is bounded and continuous²⁾ with respect to u . The range of T lies in B_2 .

¹⁾ See, for example, E. Hille and R. S. Phillips [4], p. 40, or N. I. Achieser and I. M. Glasmann [14], p. 82.

²⁾ We don't require that $T'_{(u)} k$ is continuous with respect to k .

Let T_0 be any operator on $D_0 \supset D$ into B_2 with the properties:

$$a. \quad T_0 u_0 = \theta \quad \text{for some } u_0 \in D. \quad (9.2)$$

b. T_0 has a derivative $T'_{0(u)}$ in D satisfying the same conditions as $T'_{(u)}$

c. The operators

$$T_\lambda = (1 - \lambda) T_0 + \lambda T, \quad 0 < \lambda < 1,$$

are closed.

Denote

$$U = \{u: T_\lambda u = \theta, \quad 0 \leq \lambda < 1\}.$$

Then either (9.1) has a solution or¹⁾ the sets

$$S = \{s: s = \frac{\|k\|}{\|T'_{\lambda(u)} k\|}, \quad k \in B_1, \quad u \in U, \quad 0 \leq \lambda < 1\}, \quad (9.3)$$

and

$$V = \{v: v = \|(T - T_0)u\|, \quad u \in U\}, \quad (9.4)$$

are not both bounded.

Proof. Let A be the set of all λ in $0 \leq \lambda \leq 1$ for which the equation $T_\lambda u = \theta$ has a solution. Then $A \neq \emptyset$ because $0 \in A$. Let S be bounded:

$$s \leq C_1 \quad \text{or} \quad \|T'_{\lambda(u)} k\| \geq \frac{1}{C_1} \|k\|, \quad \frac{1}{C_1} > 0.$$

Therefore²⁾, the operator $T'_{\lambda(u)}$ has a bounded inverse $T'^{-1}_{\lambda(u)}$ and

$$\|T'^{-1}_{\lambda(u)}\| \leq C_1. \quad (9.5)$$

Hence the assumptions of Theorem 7.1, supplement 7.1 b, are satisfied. Therefore, it follows that the set A is open with respect to $[0, 1]$.

Moreover, Theorem 8.1 says that each "point" $(T_\lambda, u(T_\lambda))$, $u \in U$, has an Ω -neighborhood in which $u = u(T)$ is unique, continuous and differentiable if assumption A of Chapter 8 is satisfied. This is obviously true if we restrict ourselves to

¹⁾ The statements shall not exclude each other, i.e. at least one of them is true.

²⁾ See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

$T_\lambda \in \Omega$. Then the operator $\tilde{T}'_{(u)}$ in (8.3) becomes $T'_{\lambda(u)}$. From this it follows that we can construct a unique and continuously differentiable function $\varphi(\lambda) = u(T_\lambda) \in D$ with $T_\lambda \varphi(\lambda) = \theta$ defined on some interval $0 \leq \lambda < \tilde{\lambda}$ if we apply the Theorems 7.1 and 8.1 repeatedly. Let $[0, \tilde{\lambda}]$ be the largest interval for which $\varphi(\lambda)$ can be defined by this construction under the assumption that (9.1) is not solvable, i.e. $1 \notin A$. Obviously $0 < \tilde{\lambda} < 1$ and $\tilde{\lambda} \notin A$.

Then by (8.7) we have

$$\varphi'(\lambda) = -T_{\lambda(\varphi(\lambda))}^{-1} T'_{(\lambda)(\varphi(\lambda))} = -T_{\lambda(\varphi(\lambda))}^{-1} (T - T_0) u(T_\lambda) \quad (9.6)$$

for $0 \leq \lambda < \tilde{\lambda}$. And $\varphi'(\lambda)$ is a bounded linear operator on R^1 into B_1 .

Now let $\lambda_\nu < \tilde{\lambda}$, $\nu = 1, 2, \dots$, be a sequence converging to $\tilde{\lambda}$ and $u_\nu = u(T_{\lambda_\nu}) = \varphi(\lambda_\nu)$ be the solutions of $T_{\lambda_\nu} u = \theta$ as just obtained. Then by the mean value theorem of the differential calculus we have, for $\lambda_\mu > \lambda_\nu$,

$$\|u_\nu - u_\mu\| \leq \sup_{\lambda_\nu \leq \lambda \leq \lambda_\mu} \|\varphi'(\lambda)\| |\lambda_\mu - \lambda_\nu|.$$

If we assume that the sets S and V in (9.3), (9.4), respectively, are bounded with bounds C_1 and C_2 then by (9.5) and (9.6)

$$\|u_\nu - u_\mu\| \leq C_1 C_2 |\lambda_\mu - \lambda_\nu|, \quad \mu, \nu = 1, 2, \dots$$

Hence $\{u_\nu\}$ is a Cauchy sequence and by the completeness of B_1 there exists a limit element $\tilde{u} \in B_1$:

$$\tilde{u} = \lim_{\nu \rightarrow \infty} u_\nu.$$

Because $u_\nu \in D$ and $T_{\lambda_\nu} u_\nu = \theta$, $\nu = 1, 2, \dots$, we have

$$\begin{aligned} \|T_{\tilde{\lambda}} u_\nu\| &= \|(T_{\tilde{\lambda}} - T_{\lambda_\nu}) u_\nu\| = \|(\tilde{\lambda} - \lambda_\nu)(T - T_0) u_\nu\| \\ &\leq |\tilde{\lambda} - \lambda_\nu| \|(T - T_0) u_\nu\|. \end{aligned}$$

By (9.4) and $\lambda_\nu \rightarrow \tilde{\lambda}$, $\nu \rightarrow \infty$, we have

$$\|T_{\tilde{\lambda}} u_\nu\| \rightarrow 0 \quad \text{for } u_\nu \in D, u_\nu \rightarrow \tilde{u}.$$

Since $T_{\tilde{\lambda}}$ is closed, then

$$\tilde{u} \in D \quad \text{and} \quad T_{\tilde{\lambda}} \tilde{u} = \theta.$$

Therefore, A also is closed with respect to $[0, 1]$. Thus $A = [0, 1]$ which completes the proof.

If we choose, in particular, $T_0 u = Tu - Tu_0$ for some fixed $u_0 \in D$, we get

$$T_{\lambda} u = Tu - (1 - \lambda) Tu_0 \quad \text{and} \quad T - T_0 = Tu_0 = \text{const.} \quad (9.7)$$

Thus, all assumptions on T_0 and also the boundedness of the set V are satisfied automatically, and we have the

Corollary 9.1. Assume T is a closed operator defined on an (open) domain $D \subset B_1$ and with range in B_2 . Let T have a derivative $T'_{(u)}$ there such that $T'_{(u)} - T'_{(v)}$ is a bounded operator depending continuously on $u, (u, v \in D)$.

Then either (9.1) has a solution or the set S in (9.3) is not bounded.

The condition of the boundedness of the set S is equivalent to the condition

$$\inf \{ \| T'_{\lambda(u)} k \| : \| k \| = 1, \quad k \in B_1, \quad u \in U, \quad 0 \leq \lambda < 1 \} \\ = m > 0. \quad (9.8)$$

Since $\lambda = 0$ is not excluded there is no statement if $T'_{0(u)} k$ is θ for some k ; for example, if T_0 is constant. As (9.8) or the boundedness of S is equivalent¹⁾ also to the existence of a bounded inverse of $T'_{\lambda(u)}$ the existence of a solution of (9.1) can only fail if $T'_{\lambda(u)}$ fails to exist as a bounded operator for some $\lambda \in [0, 1]$. The proof of Theorem 9.1 shows that we even can restrict ourselves to examine only $T'^{-1}_{\lambda(u)}$ for $u = \varphi(\lambda)$ or according to formula (8.6) to $(T'^{-1}_{\lambda})_{(\theta)} = (T'_{\lambda(\varphi(\lambda))})^{-1}$. Thus, writing (9.1) in the form

$$Tu = w_1, \quad (9.9)$$

and choosing $T_0 u = Tu - w_0$, $w_0 = Tu_0$, as for (9.7), we get $T_{\lambda} u = Tu - w_0 - \lambda(w_1 - w_0)$ and we have the

¹⁾ See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

Corollary 9.2. The equation (9.9) with T satisfying the assumptions of Theorem 9.1 has at least one solution if for at least one $u_0 \in D$, with $\varphi(\lambda)$ the same as in the proof of Theorem 9.1, and

$$w(\lambda) = w_0 + \lambda(w_1 - w_0), \quad (9.10)$$

the operators

$$(T'_{(\varphi(\lambda))})^{-1} = (T^{-1})'_{(w(\lambda))}, \quad 0 \leq \lambda < 1,$$

exist and are bounded uniformly in λ , or equivalently, if $T'^{-1}_{(u_0)}$ exists as a bounded operator and

$$\|(T'_{(\varphi(\lambda))})^{-1}\| = \|(T^{-1})'_{(w(\lambda))}\|,$$

remains finite with increasing λ from 0 to 1.

Example. It is well known that the equation

$$Tz \equiv \tan z = \omega, \quad z, \omega \text{ complex numbers,}$$

is not solvable only for $\omega = \pm i$. Theorem 9.1 immediately shows that the equation is solvable for all $\omega \neq \pm i$. For

$$(T^{-1})'_{(w)} = \frac{1}{1 + w^2},$$

and, with $\omega_{0_1} = 0 = \tan 0$ and $\omega_{0_2} = 1 = \tan \frac{\pi}{4}$, all points of the complex number plane can be reached on straight lines (9.10) from either 0 or 1 such that $\frac{1}{1 + (\omega(\lambda))^2}$ remains bounded with the only exceptions $\omega = \pm i$.

10. COMPLETELY CONTINUOUS OPERATORS, NEIGHBORHOOD AND INVERSE FUNCTION THEOREMS.

The assumptions of the theorems can be partially weakened if the non-linear equation can be written in the form

$$u = Vu, \quad (10.1)$$

with a completely continuous operator V . Complete continuity