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$$\begin{aligned} \Delta_2 \Delta_1 T_0 &= \Delta_1 \Delta_2 T_0 \\ &= T(x_0 + h_2 + h_1) - T(x_0 + h_1) - T(x_0 + h_2) - T(x_0) \\ &= T_x(x_0 + h_1) h_2 - T_x(x_0) h_2 + \circ(h_2) \\ &= T_{xx} h_1 h_2 + \circ(h_1) + \circ(h_2), \end{aligned}$$

and

$$\Delta_i T'_{0(u_0)} = T_u(x_0 + h_i, u_0) - T_u(x_0, u_0) = T_{xu} h_i + \circ(h_i), \quad i = 1, 2.$$

Hence by (8.9) and (8.10), neglecting the terms  $\circ(h_i)$ , it results

$$\begin{aligned} \varphi''(x_0) h_2 h_1 &= -(T_u)^{-1} \{ T_{xu}(h_1 [\varphi'(x_0) h_2] + h_2 [\varphi'(x_0) h_1]) \\ &\quad + T_{xx} h_1 h_2 + T_{uu} [\varphi'(x_0) h_1] [\varphi'(x_0) h_2] \} \end{aligned}$$

where the derivatives of  $T$  are taken at the point  $(x_0, u_0)$  [e.g.  $T_u = T_u(x_0, u_0)$ ] and, for example,  $T_{xu} hk$  means that the bilinear operator  $T_{xu} = T_{xu}(x_0, u_0)$  applies to the elements  $h$  and  $k$ . Here  $\varphi'(x_0) h$  can be expressed by  $-T_u^{-1} T_x h$  according to (8.7).

## 9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR

In this chapter a method for the proof of the existence of a solution of a non-linear equation

$$Tu = \theta, \tag{9.1}$$

is introduced which may be useful in cases where  $T$  has a derivative but cannot be written in the form  $I-V$  with completely continuous operator  $V$  or in which the complete continuity of  $V$  is difficult to show.

**THEOREM 9.1.** Assume  $T$  is a closed<sup>1)</sup> operator defined on an (open) domain  $D \subset B_1$  and there has a derivative  $T'_{(u)}$  such that  $T'_{(u)} - T'_{(v)}$  ( $u, v \in D$ ) is bounded and continuous<sup>2)</sup> with respect to  $u$ . The range of  $T$  lies in  $B_2$ .

1) See, for example, E. Hille and R. S. Phillips [4], p. 40, or N.I. Achieser and I. M. Glasmann [14], p. 82.

2) We don't require that  $T'_{(u)}k$  is continuous with respect to  $k$ .

Let  $T_0$  be any operator on  $D_0 \supset D$  into  $B_2$  with the properties:

a.  $T_0 u_0 = \theta$  for some  $u_0 \in D$ . (9.2)

b.  $T_0$  has a derivative  $T'_{0(u)}$  in  $D$  satisfying the same conditions as  $T'_{(u)}$

c. The operators

$$T_\lambda = (1 - \lambda) T_0 + \lambda T, \quad 0 < \lambda < 1,$$

are closed.

Denote

$$U = \{u: T_\lambda u = \theta, 0 \leq \lambda < 1\}.$$

Then either (9.1) has a solution or<sup>1)</sup> the sets

$$S = \left\{s: s = \frac{\|k\|}{\|T'_{\lambda(u)} k\|}, k \in B_1, u \in U, 0 \leq \lambda < 1\right\}, \quad (9.3)$$

and

$$V = \{v: v = \|(T - T_0)u\|, u \in U\}, \quad (9.4)$$

are not both bounded.

*Proof.* Let  $\Lambda$  be the set of all  $\lambda$  in  $0 \leq \lambda \leq 1$  for which the equation  $T_\lambda u = \theta$  has a solution. Then  $\Lambda \neq \emptyset$  because  $0 \in \Lambda$ . Let  $S$  be bounded:

$$s \leq C_1 \quad \text{or} \quad \|T'_{\lambda(u)} k\| \geq \frac{1}{C_1} \|k\|, \quad \frac{1}{C_1} > 0.$$

Therefore<sup>2)</sup>, the operator  $T'_{\lambda(u)}$  has a bounded inverse  $T'^{-1}_{\lambda(u)}$  and

$$\|T'^{-1}_{\lambda(u)}\| \leq C_1. \quad (9.5)$$

Hence the assumptions of Theorem 7.1, supplement 7.1 b, are satisfied. Therefore, it follows that the set  $\Lambda$  is open with respect to  $[0, 1]$ .

Moreover, Theorem 8.1 says that each "point"  $(T_\lambda, u(T_\lambda))$ ,  $u \in U$ , has an  $\Omega$ -neighborhood in which  $u = u(T)$  is unique, continuous and differentiable if assumption A of Chapter 8 is satisfied. This is obviously true if we restrict ourselves to

1) The statements shall not exclude each other, i.e. at least one of them is true.

2) See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

$T_\lambda \in \Omega$ . Then the operator  $\tilde{T}'_{(u)}$  in (8.3) becomes  $T'_{\lambda(u)}$ . From this it follows that we can construct a unique and continuously differentiable function  $\varphi(\lambda) = u(T_\lambda) \in D$  with  $T_\lambda \varphi(\lambda) = \theta$  defined on some interval  $0 \leq \lambda < \tilde{\lambda}$  if we apply the Theorems 7.1 and 8.1 repeatedly. Let  $[0, \tilde{\lambda}]$  be the largest interval for which  $\varphi(\lambda)$  can be defined by this construction under the assumption that (9.1) is not solvable, i.e.  $1 \notin \Lambda$ . Obviously  $0 < \tilde{\lambda} < 1$  and  $\tilde{\lambda} \notin \Lambda$ .

Then by (8.7) we have

$$\varphi'(\lambda) = -T'_{\lambda(\varphi(\lambda))}{}^{-1} T'_{(\lambda)(\varphi(\lambda))} = -T'_{\lambda(\varphi(\lambda))}{}^{-1} (T - T_0) u(T_\lambda) \quad (9.6)$$

for  $0 \leq \lambda < \tilde{\lambda}$ . And  $\varphi'(\lambda)$  is a bounded linear operator on  $R^1$  into  $B_1$ .

Now let  $\lambda_\nu < \tilde{\lambda}$ ,  $\nu = 1, 2, \dots$ , be a sequence converging to  $\tilde{\lambda}$  and  $u_\nu = u(T_{\lambda_\nu}) = \varphi(\lambda_\nu)$  be the solutions of  $T_{\lambda_\nu} u = \theta$  as just obtained. Then by the mean value theorem of the differential calculus we have, for  $\lambda_\mu > \lambda_\nu$ ,

$$\|u_\nu - u_\mu\| \leq \sup_{\lambda_\nu \leq \lambda \leq \lambda_\mu} \|\varphi'(\lambda)\| |\lambda_\mu - \lambda_\nu|.$$

If we assume that the sets  $S$  and  $V$  in (9.3), (9.4), respectively, are bounded with bounds  $C_1$  and  $C_2$  then by (9.5) and (9.6)

$$\|u_\nu - u_\mu\| \leq C_1 C_2 |\lambda_\mu - \lambda_\nu|, \quad \mu, \nu = 1, 2, \dots$$

Hence  $\{u_\nu\}$  is a Cauchy sequence and by the completeness of  $B_1$  there exists a limit element  $\tilde{u} \in B_1$ :

$$\tilde{u} = \lim_{\nu \rightarrow \infty} u_\nu.$$

Because  $u_\nu \in D$  and  $T_{\lambda_\nu} u_\nu = \theta$ ,  $\nu = 1, 2, \dots$ , we have

$$\begin{aligned} \|T_{\tilde{\lambda}} u_\nu\| &= \|(T_{\tilde{\lambda}} - T_{\lambda_\nu}) u_\nu\| = \|(\tilde{\lambda} - \lambda_\nu)(T - T_0) u_\nu\| \\ &\leq |\tilde{\lambda} - \lambda_\nu| \|(T - T_0) u_\nu\|. \end{aligned}$$

By (9.4) and  $\lambda_\nu \rightarrow \tilde{\lambda}$ ,  $\nu \rightarrow \infty$ , we have

$$\|T_{\tilde{\lambda}} u_\nu\| \rightarrow 0 \quad \text{for } u_\nu \in D, u_\nu \rightarrow \tilde{u}.$$

Since  $T_{\tilde{\lambda}}$  is closed, then

$$\tilde{u} \in D \quad \text{and} \quad T_{\tilde{\lambda}} \tilde{u} = \theta.$$

Therefore,  $\Lambda$  also is closed with respect to  $[0, 1]$ . Thus  $\Lambda = [0, 1]$  which completes the proof.

If we choose, in particular,  $T_0 u = Tu - Tu_0$  for some fixed  $u_0 \in D$ , we get

$$T_{\lambda} u = Tu - (1 - \lambda) Tu_0 \quad \text{and} \quad T - T_0 = Tu_0 = \text{const.} \quad (9.7)$$

Thus, all assumptions on  $T_0$  and also the boundedness of the set  $V$  are satisfied automatically, and we have the

*Corollary 9.1.* Assume  $T$  is a closed operator defined on an (open) domain  $D \subset B_1$  and with range in  $B_2$ . Let  $T$  have a derivative  $T'_{(u)}$  there such that  $T'_{(u)} - T'_{(v)}$  is a bounded operator depending continuously on  $u, (u, v \in D)$ .

Then either (9.1) has a solution or the set  $S$  in (9.3) is not bounded.

The condition of the boundedness of the set  $S$  is equivalent to the condition

$$\inf \{ \| T'_{\lambda(u)} k \| : \| k \| = 1, \quad k \in B_1, \quad u \in U, \quad 0 \leq \lambda < 1 \} \\ = m > 0. \quad (9.8)$$

Since  $\lambda = 0$  is not excluded there is no statement if  $T'_{0(u)} k$  is  $\theta$  for some  $k$ ; for example, if  $T_0$  is constant. As (9.8) or the boundedness of  $S$  is equivalent<sup>1)</sup> also to the existence of a bounded inverse of  $T'_{\lambda(u)}$  the existence of a solution of (9.1) can only fail if  $T'_{\lambda(u)}$  fails to exist as a bounded operator for some  $\lambda \in [0, 1]$ . The proof of Theorem 9.1 shows that we even can restrict ourselves to examine only  $T'_{\lambda(u)}$  for  $u = \varphi(\lambda)$  or according to formula (8.6) to  $(T'_{\lambda})'_{(\theta)} = (T'_{\lambda(\varphi(\lambda))})^{-1}$ . Thus, writing (9.1) in the form

$$Tu = w_1, \quad (9.9)$$

and choosing  $T_0 u = Tu - w_0, w_0 = Tu_0$ , as for (9.7), we get  $T_{\lambda} u = Tu - w_0 - \lambda(w_1 - w_0)$  and we have the

<sup>1)</sup> See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

*Corollary 9.2.* The equation (9.9) with  $T$  satisfying the assumptions of Theorem 9.1 has at least one solution if for at least one  $u_0 \in D$ , with  $\varphi(\lambda)$  the same as in the proof of Theorem 9.1, and

$$w(\lambda) = w_0 + \lambda(w_1 - w_0), \quad (9.10)$$

the operators

$$(T'_{(\varphi(\lambda))})^{-1} = (T^{-1})'_{(w(\lambda))}, \quad 0 \leq \lambda < 1,$$

exist and are bounded uniformly in  $\lambda$ , or equivalently, if  $T'_{(u_0)}^{-1}$  exists as a bounded operator and

$$\|(T'_{(\varphi(\lambda))})^{-1}\| = \|(T^{-1})'_{(w(\lambda))}\|,$$

remains finite with increasing  $\lambda$  from 0 to 1.

*Example.* It is well known that the equation

$$Tz \equiv \tan z = \omega, \quad z, \omega \text{ complex numbers,}$$

is not solvable only for  $\omega = \pm i$ . Theorem 9.1 immediately shows that the equation is solvable for all  $\omega \neq \pm i$ . For

$$(T^{-1})'_{(w)} = \frac{1}{1 + w^2},$$

and, with  $\omega_{0_1} = 0 = \tan 0$  and  $\omega_{0_2} = 1 = \tan \frac{\pi}{4}$ , all points of the complex number plane can be reached on straight lines (9.10) from either 0 or 1 such that  $\frac{1}{1 + (\omega(\lambda))^2}$  remains bounded with the only exceptions  $\omega = \pm i$ .

## 10. COMPLETELY CONTINUOUS OPERATORS, NEIGHBORHOOD AND INVERSE FUNCTION THEOREMS.

The assumptions of the theorems can be partially weakened if the non-linear equation can be written in the form

$$u = Vu, \quad (10.1)$$

with a completely continuous operator  $V$ . Complete continuity