9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR

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$$\Delta_{2} \Delta_{1} T_{0} = \Delta_{1} \Delta_{2} T_{0}$$

$$= T(x_{0} + h_{2} + h_{1}) - T(x_{0} + h_{1}) - T(x_{0} + h_{2}) - T(x_{0})$$

$$= T_{x}(x_{0} + h_{1}) h_{2} - T_{x}(x_{0}) h_{2} + O(h_{2})$$

$$= T_{xx} h_{1} h_{2} + O(h_{1}) + O(h_{2}),$$

and

$$\Delta_{i} T_{0(u_{0})}^{'} = T_{u}(x_{0} + h_{i}, u_{0}) - T_{u}(x_{0}, u_{0}) = T_{xu} h_{i} + O(h_{i}), \quad i = 1, 2.$$

Hence by (8.9) and (8.10), neglecting the terms \circ (h_i), it results

$$\varphi''(x_0) h_2 h_1 = -(T_u)^{-1} \{ T_{xu} (h_1 [\varphi'(x_0) h_2] + h_2 [\varphi'(x_0) h_1]) + T_{xx} h_1 h_2 + T_{uu} [\varphi'(x_0) h_1] [\varphi'(x_0) h_2] \}$$

where the derivatives of T are taken at the point (x_0, u_0) [e.g. $T_u = T_u(x_0, u_0)$] and, for example, $T_{xu}hk$ means that the bilinear operator $T_{xu} = T_{xu}(x_0, u_0)$ applies to the elements h and k. Here $\varphi'(x_0)h$ can be expressed by $-T_u^{-1}T_xh$ according to (8.7).

9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR

In this chapter a method for the proof of the existence of a solution of a non-linear equation

$$Tu = \theta, (9.1)$$

is introduced which may be useful in cases where T has a derivative but cannot be written in the form I-V with completely continuous operator V or in which the complete continuity of V is difficult to show.

THEOREM 9.1. Assume T is a closed 1) operator defined on an (open) domain $D \subset B_1$ and there has a derivative $T'_{(u)}$ such that $T'_{(u)} - T'_{(v)}$ $(u, v \in D)$ is bounded and continuous 2) with respect to u. The range of T lies in B_2 .

¹⁾ See, for example, E. Hille and R. S. Phillips [4], p. 40, or N.I. Achieser and I. M. Glasmann [14], p. 82.

²⁾ We don't require that T'(u)k is continuous with respect to k.

Let T_0 be any operator on $D_0 \supset D$ into B_2 with the properties:

a.
$$T_0 u_0 = \theta$$
 for some $u_0 \in D$. (9.2)

- b. T_0 has a derivative $T_{0(u)}^{'}$ in D satisfying the same conditions as $T_{(u)}^{'}$
- c. The operators

$$T_{\lambda} = (1-\lambda) T_0 + \lambda T, \quad 0 < \lambda < 1,$$

are closed.

Denote

$$U = \{u: T_{\lambda} u = \theta, \ 0 \le \lambda < 1\}.$$

Then either (9.1) has a solution or 1) the sets

$$S = \{s : s = \frac{\|k\|}{\|T'_{\lambda(u)}k\|}, k \in B_1, u \in U, 0 \le \lambda < 1\}, (9.3)$$

and

$$V = \{v: v = \| (T - T_0) u \|, u \in U\},$$
 (9.4)

are not both bounded.

Proof. Let Λ be the set of all λ in $0 \le \lambda \le 1$ for which the equation $T_{\lambda} u = \theta$ has a solution. Then $\Lambda \ne \emptyset$ because $0 \in \Lambda$. Let S be bounded:

$$s \le C_1$$
 or $||T'_{\lambda(u)}k|| \ge \frac{1}{C_1} ||k||, \frac{1}{C_1} > 0$.

Therefore ²), the operator $T'_{\lambda(u)}$ has a bounded inverse $T'_{\lambda(u)}$ and

$$||T_{\lambda(u)}^{'-1}|| \le C_1.$$
 (9.5)

Hence the assumptions of Theorem 7.1, supplement 7.1 b, are satisfied. Therefore, it follows that the set Λ is open with respect to [0, 1].

Moreover, Theorem 8.1 says that each "point" $(T_{\lambda}, u(T_{\lambda}))$, $u \in U$, has an Ω -neighborhood in which u = u(T) is unique, continuous and differentiable if assumption A of Chapter δ is satisfied. This is obviously true if we restrict ourselves to

¹⁾ The statements shall not exclude each other, i.e. at least one of them is true.

²⁾ See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

 $T_{\lambda} \in \Omega$. Then the operator $\tilde{T}'_{(u)}$ in (8.3) becomes $T'_{\lambda(u)}$. From this it follows that we can construct a unique and continuously differentiable function $\varphi(\lambda) = u(T_{\lambda}) \in D$ with $T_{\lambda} \varphi(\lambda) = \theta$ defined on some interval $0 \le \lambda < \tilde{\lambda}$ if we apply the Theorems 7.1 and 8.1 repeatedly. Let $[0, \tilde{\lambda}]$ be the largest interval for which $\varphi(\lambda)$ can be defined by this construction under the assumption that (9.1) is not solvable, i.e. $1 \notin \Lambda$. Obviously $0 < \tilde{\lambda} < 1$ and $\tilde{\lambda} \notin \Lambda$.

Then by (8.7) we have

$$\varphi'(\lambda) = -T_{\lambda(\varphi(\lambda))}^{'-1} T_{(\lambda)(\varphi(\lambda))}^{'} = -T_{\lambda(\varphi(\lambda))}^{'-1} (T - T_0) u(T_{\lambda})$$
 (9.6)

for $0 \leq \lambda < \tilde{\lambda}$. And $\varphi'(\lambda)$ is a bounded linear operator on R^1 into B_1 .

Now let $\lambda_{\nu} < \tilde{\lambda}$, $\nu = 1, 2, ...$, be a sequence converging to $\tilde{\lambda}$ and $u_{\nu} = u$ $(T_{\lambda_{\nu}}) = \varphi$ (λ_{ν}) be the solutions of $T_{\lambda_{\nu}} u = \theta$ as just obtained. Then by the mean value theorem of the differential calculus we have, for $\lambda_{\mu} > \lambda_{\nu}$,

$$\|u_{\nu}-u_{\mu}\| \leq \sup_{\lambda_{\nu} \leq \lambda \leq \lambda_{\mu}} \|\varphi'(\lambda)\| |\lambda_{\mu}-\lambda_{\nu}|.$$

If we assume that the sets S and V in (9.3), (9.4), respectively, are bounded with bounds C_1 and C_2 then by (9.5) and (9.6)

$$||u_{\nu}-u_{\mu}|| \le C_1 C_2 |\lambda_{\mu}-\lambda_{\nu}|, \qquad \mu, \nu = 1, 2, \dots.$$

Hence $\{u_{\nu}\}$ is a Cauchy sequence and by the completeness of B_1 there exists a limit element $\tilde{u} \in B_1$:

$$\tilde{u} = \lim_{v \to \infty} u_v$$
.

Because $u_{\nu} \in D$ and $T_{\lambda_{\nu}} u_{\nu} = \theta$, $\nu = 1, 2, ...$, we have

$$\|T_{\widetilde{\lambda}} u_{\nu}\| = \|(T_{\widetilde{\lambda}} - T_{\lambda_{\nu}}) u_{\nu}\| = \|(\widetilde{\lambda} - \lambda_{\nu}) (T - T_{0}) u_{\nu}\|$$

$$\leq |\widetilde{\lambda} - \lambda_{\nu}| \|(T - T_{0}) u_{\nu}\|.$$

By (9.4) and $\lambda_{\nu} \to \tilde{\lambda}$, $\nu \to \infty$, we have

$$\|T_{\lambda}u_{\nu}\| \to 0$$
 for $u_{\nu} \in D$, $u_{\nu} \to \tilde{u}$.

Since $T_{\tilde{\lambda}}$ is closed, then

$$\widetilde{u} \in D$$
 and $T_{\widetilde{\lambda}} \widetilde{u} = \theta$.

Therefore, Λ also is closed with respect to [0, 1]. Thus $\Lambda = [0, 1]$ which completes the proof.

If we choose, in particular, $T_0 u = Tu - Tu_0$ for some fixed $u_0 \in D$, we get

$$T_{\lambda} u = Tu - (1 - \lambda) Tu_0$$
 and $T - T_0 = Tu_0 = \text{const.}$ (9.7)

Thus, all assumptions on T_0 and also the boundedness of the set V are satisfied automatically, and we have the

Corollary 9.1. Assume T is a closed operator defined on an (open) domain $D \subset B_1$ and with range in B_2 . Let T have a derivative $T'_{(u)}$ there such that $T'_{(u)} - T'_{(v)}$ is a bounded operator depending continuously on u, $(u, v \in D)$.

Then either (9.1) has a solution or the set S in (9.3) is not bounded.

The condition of the boundedness of the set S is equivalent to the condition

inf
$$\{ \| T'_{\lambda(u)} k \| : \| k \| = 1, k \in B_1, u \in U, 0 \le \lambda < 1 \}$$

= $m > 0.$ (9.8)

Since $\lambda = 0$ is not excluded there is no statement if $T_{0(u)}$ k is θ for some k; for example, if T_0 is constant. As (9.8) or the boundedness of S is equivalent 1) also to the existence of a bounded inverse of $T'_{\lambda(u)}$ the existence of a solution of (9.1) can only fail if $T'^{-1}_{\lambda(u)}$ fails to exist as a bounded operator for some $\lambda \in [0, 1]$. The proof of Theorem 9.1 shows that we even can restrict ourselves to examine only $T'^{-1}_{\lambda(u)}$ for $u = \varphi(\lambda)$ or according to formula (8.6) to $(T^{-1}_{\lambda})'_{(\theta)} = (T'_{\lambda(\varphi(\lambda))})^{-1}$. Thus, writing (9.1) in the form

$$Tu = w_1, (9.9)$$

and choosing $T_0 u = Tu - w_0$, $w_0 = Tu_0$, as for (9.7), we get $T_{\lambda} u = Tu - w_0 - \lambda (w_1 - w_0)$ and we have the

¹⁾ See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

Corollary 9.2. The equation (9.9) with T satisfying the assumptions of Theorem 9.1 has at least one solution if for at least one $u_0 \in D$, with $\varphi(\lambda)$ the same as in the proof of Theorem 9.1, and

$$w(\lambda) = w_0 + \lambda (w_1 - w_0),$$
 (9.10)

the operators

$$\left(T_{\left(\varphi\left(\lambda\right)\right)}^{'}\right)^{-1} = \left(T^{-1}\right)_{\left(w\left(\lambda\right)\right)}^{'}, \quad 0 \leq \lambda < 1,$$

exist and are bounded uniformly in λ , or equivalently, if $T_{(u_0)}^{'-1}$ exists as a bounded operator and

$$\|(T_{(\varphi(\lambda))}^{'})^{-1}\| = \|(T^{-1})_{(w(\lambda))}^{'}\|,$$

remains finite with increasing λ from 0 to 1.

Example. It is well known that the equation

$$Tz \equiv \tan z = w$$
, z, w complex numbers,

is not solvable only for $w = \pm i$. Theorem 9.1 immediately shows that the equation is solvable for all $w \neq \pm i$. For

$$(T^{-1})'_{(w)} = \frac{1}{1+w^2},$$

and, with $w_{0_1} = 0 = \tan 0$ and $w_{0_2} = 1 = \tan \frac{\pi}{4}$, all points of the complex number plane can be reached on straight lines (9.10) from either 0 or 1 such that $\frac{1}{1 + (w(\lambda))^2}$ remains bounded with the only exceptions $w = \pm i$.

10. Completely continuous operators, NEIGHBORHOOD AND INVERSE FUNCTION THEOREMS.

The assumptions of the theorems can be partially weakened if the non-linear equation can be written in the form

$$u = Vu, (10.1)$$

with a completely continuous operator V. Complete continuity