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**Autor:** Ehrmann, Hans H.  
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This supplement covers differential operators, for example, which usually are not continuous but have a continuous inverse. For such differential operators which have a derivative satisfying the assumptions  $a)$  and  $b')$  or  $b'')$  the existence of an  $\Omega$ -neighborhood can only fail at a “ point ”  $(T, u)$  where  $T_{(u)}'^{-1}$  does not exist as a bounded linear operator. But the existence of a bounded inverse  $T_{(u)}'^{-1}$  for each  $u \in B_1$ ,  $T$  being defined everywhere in  $B_1$ , is not sufficient to insure that  $T$  has an inverse nor that the equation  $Tu = \omega$  is solvable for all  $\omega \in B_2$ .

## 8. ON THE DIFFERENTIABILITY OF THE SOLUTION.

In virtue of Theorem 7.1 and supplements the equation  $Tu = \theta$  is equivalent to  $u = u(T)$  in an  $\Omega$ -neighborhood of  $(T_0, u_0)$  under the above conditions or, in other words,  $u(T)$  is a unique function of  $T$  defined in  $\Omega$  by  $Tu = \theta$ . The conditions yield also the continuity of  $u(T)$  in the sense that  $u(T)$  tends to  $u_0$  as  $\|Tu_0\| \rightarrow 0$  or, more precisely,  $\|u(T) - u(T_0)\| \leq C \|Tu_0\|$  for some constant  $C$ . Therefore,

$$g(u) = \circ(\|u - u_0\|) \text{ implies } g(u) = \circ(\|Tu_0\|), \quad (8.1)$$

for these solutions  $u = u(T)$  of  $Tu = \theta$ .

In order to get the continuity it is sufficient essentially that  $\Delta T = T - T_0$  tends to zero at the single point  $u_0$ . But for the purpose of calculating a Fréchet-derivative of  $u(T)$  we have to know what the behaviour of  $T$  is in a neighborhood of  $u_0$  as  $\|Tu_0\| = \|\Delta Tu_0\| \rightarrow 0$ . According to the definition of the derivative we are looking for a linear operator  $L$  such that the expression

$$u(T_0 + \Delta T) - u(T_0) - L\Delta T,$$

tends to zero faster than of order one as  $\Delta T \rightarrow 0$  in a certain sense. But if we state the formula

$$\begin{aligned} u(T) - u(T_0) &= -T_{0(u_0)}'^{-1} \Delta Tu + \circ(\|u - u_0\|) \quad (8.2) \\ &= +T_{0(u_0)}'^{-1} T_0 u + \circ(\|u - u_0\|), \end{aligned}$$

which follows from

$$T_0 u - T_0 u_0 - T'_{(u_0)}(u - u_0) = \circ(\|u - u_0\|),$$

observing that  $T_0 u_0 = \theta$  and  $Tu = \theta$ , we get the difficulty that normally  $u(T)$  and  $T_0 u$  don't depend linearly on  $Tu_0$  or, equivalently,  $\circ(\|u - u_0\|)$  is not  $\circ(\|\Delta Tu\|)$  in general.

Therefore, we make the following natural assumption:

A. We assume that all operators  $T$  are differentiable at the point  $u_0$  and that  $T'_{(u_0)}$  tends to an operator  $\tilde{T}'_{(u_0)}$  for  $\|Tu_0\| \rightarrow 0$  such that

$$(\tilde{T}'_{(u_0)} - T'_{(u_0)})(u - u_0) = \circ(\|Tu_0\|) \quad \text{for } u = u(T), u_0 = u(T_0) \quad (8.3)$$

and  $\tilde{T}'_{(u_0)}$  has a bounded inverse.

The normal case is  $\tilde{T}'_{(u_0)} = T'_{(u_0)}$ , as for example in the usual implicit function theorems. A is more general.

Under this assumption we have the

**THEOREM 8.1.** Let  $T_0$  satisfy the assumptions of Theorem 7.1 and let  $\Omega$  be the  $(u_0, r, a, b)$ -neighborhood of  $T_0$  in which the equation (7.3)  $Tu = \theta$  is uniquely and continuously solvable. Furthermore, we assume that all  $T \in \Omega$  satisfy the differentiability condition A.

Then there exists a unique  $F$ -differential of the solution  $u(T)$  of (7.3) at the "point"  $T = T_0$  which has the form

$$u'(T_0) \Delta T_0 = -\tilde{T}'_{(u_0)}^{-1} \Delta T_0 u_0, \quad (8.4)$$

where

$$u_0 = u(T_0) \quad \text{and} \quad \Delta T_0 u_0 = (T - T_0) u_0 = Tu_0.$$

*Proof.* By definition of the  $F$ -differential of  $T$ ,

$$\begin{aligned} \Delta T_0 u_0 &= Tu_0 = Tu - T'_{(u_0)}(u - u_0) + \circ(\|u - u_0\|) \\ &= -T'_{(u_0)}(u - u_0) + \circ(\|u - u_0\|), \end{aligned}$$

because  $Tu = \theta$ . Hence it follows by (8.3) and (8.1) that

$$\Delta T_0 u_0 = -\tilde{T}'_{(u_0)}(u - u_0) + \circ(\|Tu_0\|),$$

or because of the existence of a bounded inverse that

$$u(T) - u(T_0) + \tilde{T}'_{(u_0)}^{-1} \Delta T_0 u_0 = \circ(\|\Delta T_0 u_0\|), \quad (8.5)$$

which implies (8.4) by definition of the  $F$ -differential.

There cannot be more than one such derivative. For let  $L_1$  and  $L_2$  be two linear operators satisfying (8.5). It results from (8.5) with  $\lambda \Delta T_0 u_0$  (for fixed  $\Delta T_0 u_0$  and real  $\lambda$ ) instead of  $\Delta T_0 u_0$

$$\|(L_1 - L_2) \Delta T_0 u_0\| = \varphi(\lambda) \quad \text{with} \quad \varphi(\lambda) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow 0,$$

which implies  $L_1 = L_2$ . This completes the proof.

In the special case  $Tu = T^*u - \omega$ ,  $T_0 u = T^*u - \omega_0$  and  $T^*u_0 = \omega_0$  the condition  $A$  is satisfied with  $\tilde{T}'_{(u_0)} = T'_{0(u_0)}$  because of  $T'_{(u_0)} = T'_{0(u_0)}$  and assumption  $b)$  of Theorem 7.1. By writing again  $T$  for  $T^*$  we get the following inverse function theorem as a corollary:

**THEOREM 8.2.** *a)* Let  $T$  be defined on the sphere  $S_0 = S(u_0, r_0) \subset B_1$  and let

$$Tu_0 = \omega_0.$$

Furthermore, let the assumptions *b)* and *c)* of Theorem 7.1 be satisfied.

Then  $T$  has a local inverse  $T^{-1}$  defined in a neighborhood of  $\omega_0$  and  $T^{-1}$  has a bounded derivative at the point  $\omega_0$ :

$$\begin{aligned} u(w) &= T^{-1}w, & u(w_0) &= T^{-1}\omega_0, \\ u'(w_0) \Delta w &= (T^{-1})'_{(w_0)} \Delta w = (T'_{(u_0)})^{-1} \Delta w, \end{aligned} \quad (8.6)$$

with  $\Delta w = w - \omega_0$ .

In these theorems it is not required that  $T$  and  $T'_{(u_0)}$  are continuous although a continuous derivative of the inverse function is asserted. Thus certain differential operators like  $F(x, \lambda, u, u', \dots, u^{(r)})$  plus certain conditions can be treated.

In the special case of an equation

$$Tu \equiv T(x)u \equiv T(x, u) = \theta, \quad T_0 u_0 = T(x_0, u_0) = \theta,$$

with  $x, u, Tu$  in Banach spaces we get the usual implicit function theorem with

$$u(T) = u(T(x)) = \varphi(x), \quad u(T_0) = \varphi(x_0),$$

if we assume that there are  $F$ -differentials  $T'_{(u)}(x) k$ , continuous in a neighborhood of  $(x_0, u_0)$  and with bounded operator  $T'^{-1}_{(u_0)}(x_0)$ , and  $T'_{(x_0)}(u_0) h$ . Then

$$\tilde{T}'_{(u_0)} = T'_{(u_0)}(x) \quad \text{and} \quad \varphi'(x_0) h = u'(T) T'_{(x_0)}(u_0) h,$$

and there results the well known formula

$$\varphi'(x_0) = - T'^{-1}_{(u_0)} \cdot T'_{(x_0)}(u_0). \quad (8.7)$$

In order to calculate the second  $F$ -differential of the solution  $u(T)$  of the equation  $Tu = \theta$  at  $T = T_0$  we assume that  $T$  has a first and a second  $F$ -derivative (with respect to  $u$ ) which are continuous<sup>1)</sup> in a neighborhood of  $u_0$ . Then also  $u'(T)$  is continuous "around  $T_0$ ", i.e. for fixed  $h = \Delta^* T$

$$\| u'(T_0 + \Delta T_0) h - u'(T_0) h \| \rightarrow 0 \quad \text{as} \quad \| \Delta T_0 u_0 \| \rightarrow 0.$$

Furthermore, according to the case when the operator  $T$  depends on the elements of a Banach space  $B_3$ , i.e.  $Tu = T(x)u$ ,  $x \in B_3$ , where  $\Delta Tu = T(x+h)u - T(x)u$ , we define  $\Delta$  to be a linear operation:

$$\Delta(T_1 + T_2)u = \Delta T_1 u + \Delta T_2 u, \quad \Delta(\lambda Tu) = \lambda \Delta Tu.$$

Then

$$\Delta_1(T + \Delta_2 T)u = \Delta_1 Tu + \Delta_1 \Delta_2 Tu,$$

and  $\Delta_1 \Delta_2 Tu$  is linear in  $\Delta_1$  and  $\Delta_2$ .

With these natural assumptions the calculation of the second order  $F$ -derivative as a bilinear operator is a straight-forward derivation. We use the formula

$$\Delta_1 Tu(T) + T'_{(u)} u'(T) \Delta_1 T = \theta, \quad (8.8)$$

at the "points"  $T = T_0$  and  $T = T_0 + \Delta_2 T_0$  and take the

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<sup>1)</sup> Less would suffice here, see below.

difference of the two expressions retaining only those terms which are linear in  $\Delta_2$ . For the sake of brevity we use the following abbreviations:

$$u_0 = u(T_0), \quad T = T_0 + \Delta_2 T_0, \quad u = u(T) = u(T_0 + \Delta_2 T_0), \\ \circ_2 = \circ(\|\Delta_2 T_0 u_0\|).$$

Then we have

$$u(T) = u_0 + u'(T_0) \Delta_2 T_0 + \circ_2, \\ k = u'(T) \Delta_1 T = u'(T_0 + \Delta_2 T_0) (\Delta_1 T_0 + \Delta_1 \Delta_2 T_0) \\ = u'(T_0 + \Delta_2 T_0) \Delta_1 T_0 + u'(T_0) \Delta_1 \Delta_2 T_0 + \circ_2, \\ \Delta_1 T u(T) - \Delta_1 T_0 u_0 = \Delta_1 T_0 u + \Delta_1 \Delta_2 T_0 u - \Delta_1 T_0 u_0 \\ = \Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_1 \Delta_2 T_0 u_0 + \circ_2, \quad \text{and} \\ T'_{0(u)} k = T'_{0(u_0)} k + T''_{0(u_0)} [u'(T_0) \Delta_2 T_0] [u'(T_0) \Delta_1 T_0] + \circ_2.$$

Hence

$$T'_{(u)} u'(T) \Delta_1 T = (T_0 + \Delta_2 T_0)'_{(u(T_0 + \Delta_2 T_0))} u'(T_0 + \Delta_2 T_0) \Delta_1 (T_0 + \\ \Delta_2 T_0) = [T'_{0(u)} + (\Delta_2 T_0)'_{(u)}] k = T'_{0(u)} k + \Delta_2 T'_{0(u_0)} k \\ = T'_{0(u)} k + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0 + \circ_2.$$

Therefore, by (8.8)

$$\theta = \Delta_1 T u + T'_{(u)} u'(T) \Delta_1 T - \Delta_1 T_0 u_0 - T'_{0(u_0)} u'(T_0) \Delta_1 T_0 \\ = \Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_1 \Delta_2 T_0 u_0 + T'_{0(u_0)} [u'(T) \Delta_1 T_0 \\ - u'(T_0) \Delta_1 T_0] + T'_{0(u_0)} u'(T_0) \Delta_1 \Delta_2 T_0 + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0 \\ + T''_{0(u_0)} [u'(T_0) \Delta_2 T_0] [u'(T_0) \Delta_1 T_0] + \circ_2.$$

If we assume as above that  $T'_{0(u_0)}$  has a bounded inverse we finally get

$$u'(T_0 + \Delta_2 T_0) \Delta_1 T_0 - u'(T_0) \Delta_1 T_0 \\ + T'_{0(u_0)}^{-1} \{ \Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0 \\ + \Delta_1 \Delta_2 T_0 u_0 + T'_{0(u_0)} u'(T_0) \Delta_1 \Delta_2 T_0 + T''_{0(u_0)} [u'(T_0) \Delta_2 T_0] \\ [u'(T_0) \Delta_1 T_0] \} + \circ(\|\Delta_2 T_0 u_0\|).$$

Therefore, the second order differential of the solution  $u(T)$  of  $Tu = \theta$  is given by

$$\begin{aligned} u''(T_0) \Delta_2 T_0 \Delta_1 T_0 &= -T_{0(u_0)}'^{-1} \{ \Delta_1 T_{0(u_0)}' u'(T_0) \Delta_2 T_0 \\ &\quad + \Delta_2 T_{0(u_0)}' u'(T_0) \Delta_1 T_0 + \Delta_1 \Delta_2 T_0 u_0 + T_{0(u_0)}'' \} \\ &\quad [u'(T_0) \Delta_2 T_0] [u'(T_0) \Delta_1 T_0] \} - u'(T_0) \Delta_1 \Delta_2 T_0. \end{aligned} \quad (8.9)$$

Here

$$u'(T_0) \Delta T_0 = -T_{0(u_0)}'^{-1} \Delta T_0 u_0.$$

It is obvious that instead of the boundedness of  $T_{0(u_0)}'$  the weaker condition  $A$  with  $\tilde{T}_{(u_0)} = T_{0(u_0)}'$  and  $Tu_0 = \Delta_2 T_0 u_0$  is sufficient for the existence of a differential of second order given by the formula (8.9). The considerations also show the existence of an  $F$ -derivative of  $n$ -th order and how to calculate it if  $T$  has  $F$ -derivatives up to the order  $n$  which are continuous in a neighborhood of  $u_0$  with the possible exception that  $T_{(u_0)}'$  satisfies condition  $A$  instead of the continuity condition. The uniqueness of the second order derivative can be shown as in the case of the first order derivative.

*Example.* For the special case

$$Tu \equiv T(x)u \equiv T(x, u) = \theta, \quad T_0 u \equiv T(x_0, u), \quad T_0 u_0 = \theta,$$

we now write  $T_x(x, u)$ ,  $T_u(x, u)$ ,  $T_{xx}(x, u)$  etc. for  $T_{(x)}'$ ,  $T_{(u)}'$ ,  $T_{(x)}''$  respectively in accordance with the usual notation of partial derivatives of a function of more than one variable.<sup>1)</sup>

Assuming  $x$ ,  $u$ ,  $T(x, u)$  to be elements of Banach spaces we have with

$$u(T) = u(T(x)) = \varphi(x),$$

the expressions

$$\varphi'(x)h = u'(T)T_x h,$$

and

$$\varphi''(x)h_2 h_1 = u''(T(x))(T_x h_2)(T_x h_1) + u'(T(x))T_{xx} h_2 h_1, \quad (8.10)$$

where the differentials are supposed to be Fréchet-differentials.

Furthermore, we have

$$\Delta_i T_0 = T(x_0 + h_i) - T(x_0) = T_x(x_0)h_i + \circ(h_i), \quad i = 1, 2,$$

<sup>1)</sup> The previous notation, however, seems to be more usual in functional analysis

$$\begin{aligned}
 \Delta_2 \Delta_1 T_0 &= \Delta_1 \Delta_2 T_0 \\
 &= T(x_0 + h_2 + h_1) - T(x_0 + h_1) - T(x_0 + h_2) - T(x_0) \\
 &= T_x(x_0 + h_1) h_2 - T_x(x_0) h_2 + \mathcal{O}(h_2) \\
 &= T_{xx} h_1 h_2 + \mathcal{O}(h_1) + \mathcal{O}(h_2),
 \end{aligned}$$

and

$$\Delta_i T'_{0(u_0)} = T_u(x_0 + h_i, u_0) - T_u(x_0, u_0) = T_{xu} h_i + \mathcal{O}(h_i), \quad i = 1, 2.$$

Hence by (8.9) and (8.10), neglecting the terms  $\mathcal{O}(h_i)$ , it results

$$\begin{aligned}
 \varphi''(x_0) h_2 h_1 &= -(T_u)^{-1} \{ T_{xu} (h_1 [\varphi'(x_0) h_2] + h_2 [\varphi'(x_0) h_1]) \\
 &\quad + T_{xx} h_1 h_2 + T_{uu} [\varphi'(x_0) h_1] [\varphi'(x_0) h_2] \}
 \end{aligned}$$

where the derivatives of  $T$  are taken at the point  $(x_0, u_0)$  [e.g.  $T_u = T_u(x_0, u_0)$ ] and, for example,  $T_{xu} hk$  means that the bilinear operator  $T_{xu} = T_{xu}(x_0, u_0)$  applies to the elements  $h$  and  $k$ . Here  $\varphi'(x_0) h$  can be expressed by  $-T_u^{-1} T_x h$  according to (8.7).

## 9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR

In this chapter a method for the proof of the existence of a solution of a non-linear equation

$$Tu = \theta, \quad (9.1)$$

is introduced which may be useful in cases where  $T$  has a derivative but cannot be written in the form  $I - V$  with completely continuous operator  $V$  or in which the complete continuity of  $V$  is difficult to show.

**THEOREM 9.1.** Assume  $T$  is a closed<sup>1)</sup> operator defined on an (open) domain  $D \subset B_1$  and there has a derivative  $T'_{(u)}$  such that  $T'_{(u)} - T'_{(v)}$  ( $u, v \in D$ ) is bounded and continuous<sup>2)</sup> with respect to  $u$ . The range of  $T$  lies in  $B_2$ .

<sup>1)</sup> See, for example, E. Hille and R. S. Phillips [4], p. 40, or N. I. Achieser and I. M. Glasmann [14], p. 82.

<sup>2)</sup> We don't require that  $T'_{(u)} k$  is continuous with respect to  $k$ .