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This supplement covers differential operators, for example, which usually are not continuous but have a continuous inverse. For such differential operators which have a derivative satisfying the assumptions $a)$ and $b')$ or $b'')$ the existence of an Ω -neighborhood can only fail at a "point" (T, u) where $T'_{(u)}^{-1}$ does not exist as a bounded linear operator. But the existence of a bounded inverse $T'_{(u)}^{-1}$ for each $u \in B_1$, T being defined everywhere in B_1 , is not sufficient to insure that T has an inverse nor that the equation $Tu = \omega$ is solvable for all $\omega \in B_2$.

8. ON THE DIFFERENTIABILITY OF THE SOLUTION.

In virtue of Theorem 7.1 and supplements the equation $Tu = \theta$ is equivalent to $u = u(T)$ in an Ω -neighborhood of (T_0, u_0) under the above conditions or, in other words, $u(T)$ is a unique function of T defined in Ω by $Tu = \theta$. The conditions yield also the continuity of $u(T)$ in the sense that $u(T)$ tends to u_0 as $\|Tu_0\| \rightarrow 0$ or, more precisely, $\|u(T) - u(T_0)\| \leq C \|Tu_0\|$ for some constant C . Therefore,

$$g(u) = o(\|u - u_0\|) \text{ implies } g(u) = o(\|Tu_0\|), \quad (8.1)$$

for these solutions $u = u(T)$ of $Tu = \theta$.

In order to get the continuity it is sufficient essentially that $\Delta T = T - T_0$ tends to zero at the single point u_0 . But for the purpose of calculating a Fréchet-derivative of $u(T)$ we have to know what the behaviour of T is in a neighborhood of u_0 as $\|Tu_0\| = \|\Delta Tu_0\| \rightarrow 0$. According to the definition of the derivative we are looking for a linear operator L such that the expression

$$u(T_0 + \Delta T) - u(T_0) - L\Delta T,$$

tends to zero faster than of order one as $\Delta T \rightarrow 0$ in a certain sense. But if we state the formula

$$\begin{aligned} u(T) - u(T_0) &= -T'_{0(u_0)} \Delta Tu + o(\|u - u_0\|) \\ &= +T'_{0(u_0)} T_0 u + o(\|u - u_0\|), \end{aligned} \quad (8.2)$$

which follows from

$$T_0 u - T_0 u_0 - T'_{0(u_0)}(u - u_0) = o(\|u - u_0\|),$$

observing that $T_0 u_0 = \theta$ and $Tu = \theta$, we get the difficulty that normally $u(T)$ and $T_0 u$ don't depend linearly on Tu_0 or, equivalently, $o(\|u - u_0\|)$ is not $o(\| \Delta T u \|)$ in general.

Therefore, we make the following natural assumption:

A. We assume that all operators T are differentiable at the point u_0 and that $T'_{(u_0)}$ tends to an operator $\tilde{T}'_{(u_0)}$ for $\|Tu_0\| \rightarrow 0$ such that

$$(\tilde{T}'_{(u_0)} - T'_{(u_0)})(u - u_0) = o(\|Tu_0\|) \quad \text{for } u = u(T), u_0 = u(T_0) \quad (8.3)$$

and $\tilde{T}'_{(u_0)}$ has a bounded inverse.

The normal case is $\tilde{T}'_{(u_0)} = T'_{0(u_0)}$, as for example in the usual implicit function theorems. A is more general.

Under this assumption we have the

THEOREM 8.1. Let T_0 satisfy the assumptions of Theorem 7.1 and let Ω be the (u_0, r, a, b) -neighborhood of T_0 in which the equation (7.3) $Tu = \theta$ is uniquely and continuously solvable. Furthermore, we assume that all $T \in \Omega$ satisfy the differentiability condition A.

Then there exists a unique F -differential of the solution $u(T)$ of (7.3) at the "point" $T = T_0$ which has the form

$$u'(T_0) \Delta T_0 = -\tilde{T}'_{(u_0)}^{-1} \Delta T_0 u_0, \quad (8.4)$$

where

$$u_0 = u(T_0) \quad \text{and} \quad \Delta T_0 u_0 = (T - T_0)u_0 = Tu_0.$$

Proof. By definition of the F -differential of T ,

$$\begin{aligned} \Delta T_0 u_0 &= Tu_0 = Tu - T'_{(u_0)}(u - u_0) + o(\|u - u_0\|) \\ &= -T'_{(u_0)}(u - u_0) + o(\|u - u_0\|), \end{aligned}$$

because $Tu = \theta$. Hence it follows by (8.3) and (8.1) that

$$\Delta T_0 u_0 = -\tilde{T}'_{(u_0)}(u - u_0) + o(\|Tu_0\|),$$

or because of the existence of a bounded inverse that

$$u(T) - u(T_0) + \tilde{T}'_{(u_0)} \Delta T_0 u_0 = o(\|\Delta T_0 u_0\|), \quad (8.5)$$

which implies (8.4) by definition of the F -differential.

There cannot be more than one such derivative. For let L_1 and L_2 be two linear operators satisfying (8.5). It results from (8.5) with $\lambda \Delta T_0 u_0$ (for fixed $\Delta T_0 u_0$ and real λ) instead of $\Delta T_0 u_0$

$$\|(L_1 - L_2) \Delta T_0 u_0\| = \varphi(\lambda) \quad \text{with} \quad \varphi(\lambda) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow 0,$$

which implies $L_1 = L_2$. This completes the proof.

In the special case $Tu = T^*u - \omega$, $T_0 u = T^*u - \omega_0$ and $T^*u_0 = \omega_0$ the condition A is satisfied with $\tilde{T}'_{(u_0)} = T'_{0(u_0)}$ because of $T'_{(u_0)} = T'_{0(u_0)}$ and assumption $b)$ of Theorem 7.1. By writing again T for T^* we get the following inverse function theorem as a corollary:

THEOREM 8.2. $a)$ Let T be defined on the sphere $S_0 = S(u_0, r_0) \subset B_1$ and let

$$Tu_0 = w_0.$$

Furthermore, let the assumptions $b)$ and $c)$ of Theorem 7.1 be satisfied.

Then T has a local inverse T^{-1} defined in a neighborhood of w_0 and T^{-1} has a bounded derivative at the point w_0 :

$$\begin{aligned} u(w) &= T^{-1} w, \quad u(w_0) = T^{-1} w_0, \\ u'(w_0) \Delta w &= (T^{-1})'_{(w_0)} \Delta w = (T'_{(u_0)})^{-1} \Delta w, \end{aligned} \quad (8.6)$$

with $\Delta w = w - w_0$.

In these theorems it is not required that T and $T'_{(u_0)}$ are continuous although a continuous derivative of the inverse function is asserted. Thus certain differential operators like $F(x, \lambda, u, u', \dots, u^{(r)})$ plus certain conditions can be treated.

In the special case of an equation

$$Tu \equiv T(x)u \equiv T(x, u) = \theta, \quad T_0 u_0 = T(x_0, u_0) = \theta,$$

with x, u, Tu in Banach spaces we get the usual implicit function theorem with

$$u(T) = u(T(x)) = \varphi(x), \quad u(T_0) = \varphi(x_0),$$

if we assume that there are F -differentials $T'_{(u)}(x)h$, continuous in a neighborhood of (x_0, u_0) and with bounded operator $T'^{-1}_{(u_0)}(x_0)$, and $T'_{(x_0)}(u_0)h$. Then

$$\tilde{T}'_{(u_0)} = T'_{(u_0)}(x) \quad \text{and} \quad \varphi'(x_0)h = u'(T) T'_{(x_0)}(u_0)h,$$

and there results the well known formula

$$\varphi'(x_0) = -T'^{-1}_{(u_0)} \cdot T'_{(x_0)}(u_0). \quad (8.7)$$

In order to calculate the second F -differential of the solution $u(T)$ of the equation $Tu = \theta$ at $T = T_0$ we assume that T has a first and a second F -derivative (with respect to u) which are continuous¹⁾ in a neighborhood of u_0 . Then also $u'(T)$ is continuous "around T_0 ", i.e. for fixed $h = \Delta^*T$

$$\|u'(T_0 + \Delta T_0)h - u'(T_0)h\| \rightarrow 0 \quad \text{as} \quad \|\Delta T_0 u_0\| \rightarrow 0.$$

Furthermore, according to the case when the operator T depends on the elements of a Banach space B_3 , i.e. $Tu = T(x)u$, $x \in B_3$, where $\Delta Tu = T(x+h)u - T(x)u$, we define Δ to be a linear operation:

$$\Delta(T_1 + T_2)u = \Delta T_1 u + \Delta T_2 u, \quad \Delta(\lambda Tu) = \lambda \Delta Tu.$$

Then

$$\Delta_1(T + \Delta_2 T)u = \Delta_1 Tu + \Delta_1 \Delta_2 Tu,$$

and $\Delta_1 \Delta_2 Tu$ is linear in Δ_1 and Δ_2 .

With these natural assumptions the calculation of the second order F -derivative as a bilinear operator is a straight-forward derivation. We use the formula

$$\Delta_1 Tu(T) + T'_{(u)} u'(T) \Delta_1 T = \theta, \quad (8.8)$$

at the "points" $T = T_0$ and $T = T_0 + \Delta_2 T_0$ and take the

¹⁾ Less would suffice here, see below.

difference of the two expressions retaining only those terms which are linear in Δ_2 . For the sake of brevity we use the following abbreviations:

$$u_0 = u(T_0), \quad T = T_0 + \Delta_2 T_0, \quad u = u(T) = u(T_0 + \Delta_2 T_0), \\ \circ_2 = \circ(\|\Delta_2 T_0 u_0\|).$$

Then we have

$$u(T) = u_0 + u'(T_0) \Delta_2 T_0 + \circ_2,$$

$$k = u'(T) \Delta_1 T = u'(T_0 + \Delta_2 T_0) (\Delta_1 T_0 + \Delta_1 \Delta_2 T_0) \\ = u'(T_0 + \Delta_2 T_0) \Delta_1 T_0 + u'(T_0) \Delta_1 \Delta_2 T_0 + \circ_2,$$

$$\Delta_1 T u(T) - \Delta_1 T_0 u_0 = \Delta_1 T_0 u + \Delta_1 \Delta_2 T_0 u - \Delta_1 T_0 u_0 \\ = \Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_1 \Delta_2 T_0 u_0 + \circ_2, \quad \text{and}$$

$$T'_{0(u)} k = T'_{0(u_0)} k + T''_{0(u_0)} [u'(T_0) \Delta_2 T_0] [u'(T_0) \Delta_1 T_0] + \circ_2.$$

Hence

$$T'_{(u)} u'(T) \Delta_1 T = (T_0 + \Delta_2 T_0)'_{(u(T_0 + \Delta_2 T_0))} u'(T_0 + \Delta_2 T_0) \Delta_1 (T_0 + \Delta_2 T_0) \\ = [T'_{0(u)} + (\Delta_2 T_0)'_{(u)}] k = T'_{0(u)} k + \Delta_2 T'_{0(u_0)} k \\ = T'_{0(u)} k + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0 + \circ_2.$$

Therefore, by (8.8)

$$\theta = \Delta_1 T u + T'_{(u)} u'(T) \Delta_1 T - \Delta_1 T_0 u_0 - T'_{0(u_0)} u'(T_0) \Delta_1 T_0 \\ = \Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_1 \Delta_2 T_0 u_0 + T'_{0(u_0)} [u'(T) \Delta_1 T_0 \\ - u'(T_0) \Delta_1 T_0] + T'_{0(u_0)} u'(T_0) \Delta_1 \Delta_2 T_0 + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0 \\ + T''_{0(u_0)} [u'(T_0) \Delta_2 T_0] [u'(T_0) \Delta_1 T_0] + \circ_2.$$

If we assume as above that $T'_{0(u_0)}$ has a bounded inverse we finally get

$$u'(T_0 + \Delta_2 T_0) \Delta_1 T_0 - u'(T_0) \Delta_1 T_0 \\ + T'^{-1}_{0(u_0)} \{ \Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0 \\ + \Delta_1 \Delta_2 T_0 u_0 + T'_{0(u_0)} u'(T_0) \Delta_1 \Delta_2 T_0 + T''_{0(u_0)} [u'(T_0) \Delta_2 T_0] \\ [u'(T_0) \Delta_1 T_0] \} + \circ(\|\Delta_2 T_0 u_0\|).$$

Therefore, the second order differential of the solution $u(T)$ of $Tu = \theta$ is given by

$$\begin{aligned} u''(T_0) \Delta_2 T_0 \Delta_1 T_0 = & -T_{0(u_0)}'^{-1} \{ \Delta_1 T_{0(u_0)}' u'(T_0) \Delta_2 T_0 \\ & + \Delta_2 T_{0(u_0)}' u'(T_0) \Delta_1 T_0 + \Delta_1 \Delta_2 T_0 u_0 + T_{0(u_0)}'' \} \\ & [u'(T_0) \Delta_2 T_0] [u'(T_0) \Delta_1 T_0] \} - u'(T_0) \Delta_1 \Delta_2 T_0. \end{aligned} \quad (8.9)$$

Here

$$u'(T_0) \Delta T_0 = -T_{(u_0)}'^{-1} \Delta T_0 u_0.$$

It is obvious that instead of the boundedness of $T_{0(u_0)}'$ the weaker condition A with $\tilde{T}_{(u_0)}' = T_{0(u_0)}'$ and $Tu_0 = \Delta_2 T_0 u_0$ is sufficient for the existence of a differential of second order given by the formula (8.9). The considerations also show the existence of an F -derivative of n -th order and how to calculate it if T has F -derivatives up to the order n which are continuous in a neighborhood of u_0 with the possible exception that $T_{(u_0)}'$ satisfies condition A instead of the continuity condition. The uniqueness of the second order derivative can be shown as in the case of the first order derivative.

Example. For the special case

$$Tu \equiv T(x)u \equiv T(x, u) = \theta, \quad T_0 u \equiv T(x_0, u), \quad T_0 u_0 = \theta,$$

we now write $T_x(x, u)$, $T_u(x, u)$, $T_{xx}(x, u)$ etc. for $T_{(x)}'$, $T_{(u)}'$, $T_{(x)}''$ respectively in accordance with the usual notation of partial derivatives of a function of more than one variable.¹⁾

Assuming $x, u, T(x, u)$ to be elements of Banach spaces we have with

$$u(T) = u(T(x)) = \varphi(x),$$

the expressions

$$\varphi'(x)h = u'(T)T_x h,$$

and

$$\varphi''(x)h_2 h_1 = u''(T(x))(T_x h_2)(T_x h_1) + u'(T(x))T_{xx} h_2 h_1, \quad (8.10)$$

where the differentials are supposed to be Fréchet-differentials.

Furthermore, we have

$$\Delta_i T_0 = T(x_0 + h_i) - T(x_0) = T_x(x_0)h_i + o(h_i), \quad i = 1, 2,$$

¹⁾ The previous notation, however, seems to be more usual in functional analysis

$$\begin{aligned}
 \Delta_2 \Delta_1 T_0 &= \Delta_1 \Delta_2 T_0 \\
 &= T(x_0 + h_2 + h_1) - T(x_0 + h_1) - T(x_0 + h_2) + T(x_0) \\
 &= T_x(x_0 + h_1) h_2 - T_x(x_0) h_2 + o(h_2) \\
 &= T_{xx} h_1 h_2 + o(h_1) + o(h_2),
 \end{aligned}$$

and

$$\Delta_i T'_{0(u_0)} = T_u(x_0 + h_i, u_0) - T_u(x_0, u_0) = T_{xu} h_i + o(h_i), \quad i = 1, 2.$$

Hence by (8.9) and (8.10), neglecting the terms $o(h_i)$, it results

$$\begin{aligned}
 \varphi''(x_0) h_2 h_1 &= -(T_u)^{-1} \{ T_{xu} (h_1 [\varphi'(x_0) h_2] + h_2 [\varphi'(x_0) h_1]) \\
 &\quad + T_{xx} h_1 h_2 + T_{uu} [\varphi'(x_0) h_1] [\varphi'(x_0) h_2] \}
 \end{aligned}$$

where the derivatives of T are taken at the point (x_0, u_0) [e.g. $T_u = T_u(x_0, u_0)$] and, for example, $T_{xu} h k$ means that the bilinear operator $T_{xu} = T_{xu}(x_0, u_0)$ applies to the elements h and k . Here $\varphi'(x_0) h$ can be expressed by $-T_u^{-1} T_x h$ according to (8.7).

9. A GLOBAL EXISTENCE THEOREM USING THE DIFFERENTIABILITY OF THE OPERATOR

In this chapter a method for the proof of the existence of a solution of a non-linear equation

$$Tu = \theta, \tag{9.1}$$

is introduced which may be useful in cases where T has a derivative but cannot be written in the form $I - V$ with completely continuous operator V or in which the complete continuity of V is difficult to show.

THEOREM 9.1. Assume T is a closed¹⁾ operator defined on an (open) domain $D \subset B_1$ and there has a derivative $T'_{(u)}$ such that $T'_{(u)} - T'_{(v)}$ ($u, v \in D$) is bounded and continuous²⁾ with respect to u . The range of T lies in B_2 .

¹⁾ See, for example, E. Hille and R. S. Phillips [4], p. 40, or N. I. Achieser and I. M. Glasmann [14], p. 82.

²⁾ We don't require that $T'_{(u)} k$ is continuous with respect to k .