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Artikel: ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS

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This supplement covers differential operators, for example, which usually are not continuous but have a continuous inverse. For such differential operators which have a derivative satisfying the assumptions *a)* and *b')* or *b")* the existence of an Ω-neighborhood can only fail at a " point " (T, u) where $T'^{-1}_{(u)}$ does not exist as a bounded linear operator. But the existence of a bounded inverse $T'^{-1}_{(u)}$ for each $u \in B_1$, T being defined everywhere in B_1, is not sufficient to insure that T has an inverse nor that the equation $Tu = w$ is solvable for all $w \in B_2$.

8. On the differentiability of the solution.

In virtue of Theorem 7.1 and supplements the equation $Tu = \theta$ is equivalent to $u = u\,(T)$ in an Ω-neighborhood of (T_0, u_0) under the above conditions or, in other words, $u\,(T)$ is a unique function of T defined in Ω by $Tu = \theta$. The conditions yield also the continuity of $u\,(T)$ in the sense that $u\,(T)$ tends to u_0 as $\| Tu_0 \| \to 0$ or, more precisely, $\| u\,(T) - u\,(T_0) \| \leqq C \| Tu_0 \|$ for some constant C. Therefore,

$$g\,(u) = \circ(\| u - u_0 \|) \quad \text{implies} \quad g\,(u) = \circ(\| Tu_0 \|), \qquad (8.1)$$

for these solutions $u = u\,(T)$ of $Tu = \theta$.

In order to get the continuity it is sufficient essentially that $\Delta T = T - T_0$ tends to zero at the single point u_0. But for the purpose of calculating a Fréchet-derivative of $u\,(T)$ we have to know what the behaviour of T is in a neighborhood of u_0 as $\| Tu_0 \| = \| \Delta Tu_0 \| \to 0$. According to the definition of the derivative we are looking for a linear operator L such that the expression

$$u\,(T_0 + \Delta T) - u\,(T_0) - L \Delta T,$$

tends to zero faster than of order one as $\Delta T \to 0$ in a certain sense. But if we state the formula

$$\begin{aligned} u\,(T) - u\,(T_0) &= -T'^{-1}_{0(u_0)} \Delta T u + \circ(\| u - u_0 \|) \qquad (8.2) \\ &= +T'^{-1}_{0(u_0)} T_0\, u + \circ(\| u - u_0 \|), \end{aligned}$$

which follows from

$$T_0 u - T_0 u_0 - T'_{0(u_0)}(u - u_0) = \circ(\| u - u_0 \|),$$

observing that $T_0 u_0 = \theta$ and $Tu = \theta$, we get the difficulty that normally $u(T)$ and $T_0 u$ don't depend linearly on Tu_0 or, equivalently, $\circ(\| u - u_0 \|)$ is not $\circ(\| \Delta Tu \|)$ in general.

Therefore, we make the following natural assumption:

A. We assume that all operators T are differentiable at the point u_0 and that $T'_{(u_0)}$ tends to an operator $\tilde{T}'_{(u_0)}$ for $\| Tu_0 \| \to 0$ such that

$$(\tilde{T}'_{(u_0)} - T'_{(u_0)})(u - u_0) = \circ(\| Tu_0 \|) \quad \text{for} \quad u = u(T), \; u_0 = u(T_0) \tag{8.3}$$

and $\tilde{T}'_{(u_0)}$ has a bounded inverse.

The normal case is $\tilde{T}'_{(u_0)} = T'_{0(u_0)}$, as for example in the usual implicit function theorems. A is more general.

Under this assumption we have the

THEOREM 8.1. Let T_0 satisfy the assumptions of Theorem 7.1 and let Ω be the (u_0, r, a, b)-neighborhood of T_0 in which the equation (7.3) $Tu = \theta$ is uniquely and continuously solvable. Furthermore, we assume that all $T \varepsilon \Omega$ satisfy the differentiability condition *A*.

Then there exists a unique *F*-differential of the solution $u(T)$ of (7.3) at the " point " $T = T_0$ which has the form

$$u'(T_0)\Delta T_0 = -\tilde{T}'^{-1}_{(u_0)} \Delta T_0 u_0, \tag{8.4}$$

where

$$u_0 = u(T_0) \quad \text{and} \quad \Delta T_0 u_0 = (T - T_0) u_0 = Tu_0.$$

Proof. By definition of the *F*-differential of T,

$$\Delta T_0 u_0 = Tu_0 = Tu - T'_{(u_0)}(u - u_0) + \circ(\| u - u_0 \|)$$

$$= -T'_{(u_0)}(u - u_0) + \circ(\| u - u_0 \|),$$

because $Tu = \theta$. Hence it follows by (8.3) and (8.1) that

$$\Delta T_0 u_0 = -\tilde{T}'_{(u_0)}(u - u_0) + \circ(\| Tu_0 \|),$$

or because of the existence of a bounded inverse that

$$u(T)-u(T_0)+\tilde{T}'^{-1}_{(u_0)}\Delta T_0 u_0 = o(\| \Delta T_0 u_0 \|), \tag{8.5}$$

which implies (8.4) by definition of the F-differential.

There cannot be more than one such derivative. For let L_1 and L_2 be two linear operators satisfying (8.5). It results from (8.5) with $\lambda \Delta T_0 u_0$ (for fixed $\Delta T_0 u_0$ and real λ) instead of $\Delta T_0 u_0$

$$\| (L_1 - L_2) \Delta T_0 u_0 \| = \varphi(\lambda) \quad \text{with} \quad \varphi(\lambda) \to 0 \quad \text{as} \quad \lambda \to 0,$$

which implies $L_1 = L_2$. This completes the proof.

In the special case $Tu = T^*u - w$, $T_0 u = T^*u - w_0$ and $T^*u_0 = w_0$ the condition A is satisfied with $\tilde{T}'_{(u_0)} = T'_{0(u_0)}$ because of $T'_{(u_0)} = T'_{0(u_0)}$ and assumption *b)* of Theorem 7.1. By writing again T for T^* we get the following inverse function theorem as a corollary:

THEOREM 8.2. *a)* Let T be defined on the sphere $S_0 = S(u_0, r_0) \subset B_1$ and let

$$Tu_0 = w_0 .$$

Furthermore, let the assumptions *b)* and *c)* of Theorem 7.1 be satisfied.

Then T has a local inverse T^{-1} defined in a neighborhood of w_0 and T^{-1} has a bounded derivative at the point w_0:

$$u(w) = T^{-1} w, \qquad u(w_0) = T^{-1} w_0,$$

$$u'(w_0)\Delta w = (T^{-1})'_{(w_0)} \Delta w = (T'_{(u_0)})^{-1} \Delta w, \tag{8.6}$$

with $\Delta w = w - w_0$.

In these theorems it is not required that T and $T'_{(u_0)}$ are continuous although a continuous derivative of the inverse function is asserted. Thus certain differential operators like $F(x, \lambda, u, u', \ldots, u^{(r)})$ plus certain conditions can be treated.

In the special case of an equation

$$Tu \equiv T(x)u \equiv T(x,u) = \theta, \quad T_0 u_0 = T(x_0, u_0) = \theta,$$

with x, u, Tu in Banach spaces we get the usual implicit function theorem with

$$u(T) = u(T(x)) = \varphi(x), \qquad u(T_0) = \varphi(x_0),$$

if we assume that there are F-differentials $T'_{(u)}(x)\,k$, continuous in a neighborhood of (x_0, u_0) and with bounded operator $T'^{-1}_{(u_0)}(x_0)$, and $T'_{(x_0)}(u_0)\,h$. Then

$$\tilde{T}'_{(u_0)} = T'_{(u_0)}(x) \quad \text{and} \quad \varphi'(x_0)\,h = u'(T)\,T'_{(x_0)}(u_0)\,h,$$

and there results the well known formula

$$\varphi'(x_0) = -T'^{-1}_{(u_0)} \cdot T'_{(x_0)}(u_0). \tag{8.7}$$

In order to calculate the second F-differential of the solution $u(T)$ of the equation $Tu = \theta$ at $T = T_0$ we assume that T has a first and a second F-derivative (with respect to u) which are continuous[1]) in a neighborhood of u_0. Then also $u'(T)$ is continuous " around T_0 ", i.e. for fixed $h = \Delta^* T$

$$\| u'(T_0 + \Delta T_0)\,h - u'(T_0)\,h \| \to 0 \quad \text{as} \quad \| \Delta T_0 u_0 \| \to 0.$$

Furthermore, according to the case when the operator T depends on the elements of a Banach space B_3, i.e. $Tu = T(x)\,u$, $x \in B_3$, where $\Delta Tu = T(x+h)\,u - T(x)\,u$, we define Δ to be a linear operation:

$$\Delta(T_1 + T_2)\,u = \Delta T_1 u + \Delta T_2 u, \quad \Delta(\lambda Tu) = \lambda \Delta Tu.$$

Then

$$\Delta_1 (T + \Delta_2 T)\,u = \Delta_1 Tu + \Delta_1 \Delta_2 Tu,$$

and $\Delta_1 \Delta_2 Tu$ is linear in Δ_1 and Δ_2.

With these natural assumptions the calculation of the second order F-derivative as a bilinear operator is a straight-forward derivation. We use the formula

$$\Delta_1 Tu(T) + T'_{(u)}\,u'(T)\,\Delta_1 T = \theta, \tag{8.8}$$

at the " points " $T = T_0$ and $T = T_0 + \Delta_2 T_0$ and take the

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1) Less would suffice here, see below.

difference of the two expressions retaining only those terms which are linear in Δ_2. For the sake of brevity we use the following abbreviations:

$$u_0 = u(T_0), \quad T = T_0 + \Delta_2 T_0, \quad u = u(T) = u(T_0 + \Delta_2 T_0),$$
$$\circ_2 = \circ(\| \Delta_2 T_0 u_0 \|).$$

Then we have

$$u(T) = u_0 + u'(T_0) \Delta_2 T_0 + \circ_2,$$
$$k = u'(T) \Delta_1 T = u'(T_0 + \Delta_2 T_0)(\Delta_1 T_0 + \Delta_1 \Delta_2 T_0)$$
$$= u'(T_0 + \Delta_2 T_0) \Delta_1 T_0 + u'(T_0) \Delta_1 \Delta_2 T_0 + \circ_2,$$
$$\Delta_1 T u(T) - \Delta_1 T_0 u_0 = \Delta_1 T_0 u + \Delta_1 \Delta_2 T_0 u - \Delta_1 T_0 u_0$$
$$= \Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_1 \Delta_2 T_0 u_0 + \circ_2, \quad \text{and}$$
$$T'_{0(u)} k = T'_{0(u_0)} k + T''_{0(u_0)} [u'(T_0) \Delta_2 T_0][u'(T_0) \Delta_1 T_0] + \circ_2.$$

Hence

$$T'_{(u)} u'(T) \Delta_1 T = (T_0 + \Delta_2 T_0)'_{(u(T_0 + \Delta_2 T_0))} u'(T_0 + \Delta_2 T_0) \Delta_1 (T_0 + \Delta_2 T_0) = [T'_{0(u)} + (\Delta_2 T_0)'_{(u)}] k = T'_{0(u)} k + \Delta_2 T'_{0(u_0)} k$$
$$= T'_{0(u)} k + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0 + \circ_2.$$

Therefore, by (8.8)

$$\theta = \Delta_1 T u + T'_{(u)} u'(T) \Delta_1 T - \Delta_1 T_0 u_0 - T'_{0(u_0)} u'(T_0) \Delta_1 T_0$$
$$= \Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_1 \Delta_2 T_0 u_0 + T'_{0(u_0)} [u'(T) \Delta_1 T_0 - u'(T_0) \Delta_1 T_0] + T'_{0(u_0)} u'(T_0) \Delta_1 \Delta_2 T_0 + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0$$
$$+ T''_{0(u_0)} [u'(T_0) \Delta_2 T_0][u'(T_0) \Delta_1 T_0] + \circ_2.$$

If we assume as above that $T'_{0(u_0)}$ has a bounded inverse we finally get

$$u'(T_0 + \Delta_2 T_0) \Delta_1 T_0 - u'(T_0) \Delta_1 T_0$$
$$+ T'^{-1}_{0(u_0)} \{\Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0$$
$$+ \Delta_1 \Delta_2 T_0 u_0 + T'_{0(u_0)} u'(T_0) \Delta_1 \Delta_2 T_0 + T''_{0(u_0)} [u'(T_0) \Delta_2 T_0]$$
$$[u'(T_0) \Delta_1 T_0]\} + \circ(\| \Delta_2 T_0 u_0 \|).$$

Therefore, the second order differential of the solution $u(T)$ of $Tu = \theta$ is given by

$$u''(T_0)\Delta_2 T_0 \Delta_1 T_0 = -T'^{-1}_{0(u_0)}\{\Delta_1 T'_{0(u_0)} u'(T_0)\Delta_2 T_0 + \Delta_2 T'_{0(u_0)} u'(T_0)\Delta_1 T_0 + \Delta_1 \Delta_2 T_0 u_0 + T''_{0(u_0)} [u'(T_0)\Delta_2 T_0][u'(T_0)\Delta_1 T_0]\} - u'(T_0)\Delta_1 \Delta_2 T_0 . \tag{8.9}$$

Here

$$u'(T_0)\Delta T_0 = -T'^{-1}_{(u_0)} \Delta T_0 u_0 .$$

It is obvious that instead of the boundedness of $T'_{0(u_0)}$ the weaker condition A with $\tilde{T}'_{(u_0)} = T'_{0(u_0)}$ and $Tu_0 = \Delta_2 T_0 u_0$ is sufficient for the existence of a differential of second order given by the formula (8.9). The considerations also show the existence of an F-derivative of *n-th* order and how to calculate it if T has F-derivatives up to the order n which are continuous in a neighborhood of u_0 with the possible exception that $T'_{(u_0)}$ satisfies condition A instead of the continuity condition. The uniqueness of the second order derivative can be shown as in the case of the first order derivative.

Example. For the special case

$$Tu \equiv T(x)u \equiv T(x,u) = \theta, \quad T_0 u \equiv T(x_0, u), \quad T_0 u_0 = \theta,$$

we now write $T_x(x, u)$, $T_u(x, u)$, $T_{xx}(x, u)$ etc. for $T'_{(x)}$, $T'_{(u)}$, $T''_{(x)}$ respectively in accordance with the usual notation of partial derivatives of a function of more than one variable.[1])

Assuming x, u, $T(x, u)$ to be elements of Banach spaces we have with

$$u(T) = u\big(T(x)\big) = \varphi(x),$$

the expressions

$$\varphi'(x)h = u'(T)\, T_x h,$$

and

$$\varphi''(x)h_2 h_1 = u''\big(T(x)\big)(T_x h_2)(T_x h_1) + u'\big(T(x)\big) T_{xx} h_2 h_1 , \tag{8.10}$$

where the differentials are supposed to be Fréchet-differentials.

Furthermore, we have

$$\Delta_i T_0 = T(x_0 + h_i) - T(x_0) = T_x(x_0) h_i + o(h_i), \qquad i = 1, 2,$$

1) The previous notation, however, seems to be more usual in functional analysis

$$\begin{aligned}\Delta_2 \Delta_1 T_0 &= \Delta_1 \Delta_2 T_0 \\ &= T(x_0 + h_2 + h_1) - T(x_0 + h_1) - T(x_0 + h_2) - T(x_0) \\ &= T_x(x_0 + h_1) h_2 - T_x(x_0) h_2 + \circ(h_2) \\ &= T_{xx} h_1 h_2 + \circ(h_1) + \circ(h_2),\end{aligned}$$

and

$$\Delta_i T'_{0(u_0)} = T_u(x_0 + h_i, u_0) - T_u(x_0, u_0) = T_{xu} h_i + \circ(h_i), \quad i = 1, 2.$$

Hence by (8.9) and (8.10), neglecting the terms $\circ(h_i)$, it results

$$\begin{aligned}\varphi''(x_0) h_2 h_1 &= -(T_u)^{-1} \{ T_{xu}(h_1 [\varphi'(x_0) h_2] + h_2 [\varphi'(x_0) h_1]) \\ &\quad + T_{xx} h_1 h_2 + T_{uu} [\varphi'(x_0) h_1][\varphi'(x_0) h_2] \}\end{aligned}$$

where the derivatives of T are taken at the point (x_0, u_0) [e.g. $T_u = T_u(x_0, u_0)$] and, for example, $T_{xu} hk$ means that the bilinear operator $T_{xu} = T_{xu}(x_0, u_0)$ applies to the elements h and k. Here $\varphi'(x_0) h$ can be expressed by $-T_u^{-1} T_x h$ according to (8.7).

9. A global existence theorem using the differentiability of the operator

In this chapter a method for the proof of the existence of a solution of a non-linear equation

$$Tu = \theta, \tag{9.1}$$

is introduced which may be useful in cases where T has a derivative but cannot be written in the form I-V with completely continuous operator V or in which the complete continuity of V is difficult to show.

THEOREM 9.1. Assume T is a closed[1]) operator defined on an (open) domain $D \subset B_1$ and there has a derivative $T'_{(u)}$ such that $T'_{(u)} - T'_{(v)}$ $(u, v \in D)$ is bounded and continuous[2]) with respect to u. The range of T lies in B_2.

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1) See, for example, E. Hille and R. S. Phillips [4], p. 40, or N.I. Achieser and I. M. Glasmann [14], p. 82.

2) We don't require that $T'_{(u)}k$ is continuous with respect to k.