

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 9 (1963)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS
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Kapitel: 7. DIFFERENTIABLE OPERATORS, IMPLICIT FUNCTION THEOREMS.
DOI: <https://doi.org/10.5169/seals-38780>

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assertions remain true except the last one that T is a homeomorphism of B_1 onto B_2 . If there exist two subdomains D_a and D_a^* of D' then the assumptions of Theorem 6.1 cannot hold on a whole path P in B_1 connecting D_a and D_a^* : Either T is not defined everywhere on P as a continuous operator or there does not exist an operator K with bounded inverse satisfying α), β) and γ) of Theorem 4.1.

A similar theorem can be stated using the assumptions of Theorem 4.1 a as a basis.

7. DIFFERENTIABLE OPERATORS, IMPLICIT FUNCTION THEOREMS.

If the operator T is assumed to be differentiable in the sense of Fréchet (section 2 c) then the operator $T'_{(u_0)}$ can be taken as operator K in the previous theorems and similar theorems can be stated.

THEOREM 7.1. $a)$ Let T_0 be defined on the sphere $S_0 = S(u_0, r_0) \subset B_1$ and let

$$T_0 u_0 = \theta. \quad (7.1)$$

$b)$ Let T_0 have a (not necessarily bounded) derivative $T'_{0(u_0)} = K$ at the point u_0 and let K have a bounded inverse K^{-1} defined on B_2 .

$c)$ Assume there are positive numbers $r' \leq r_0$ and $m = m(r') < \|K^{-1}\|^{-1}$ with

$$\|T_0(u_0 + u - v) - T_0 u + T_0 v\| \leq m \|u - v\|, \quad u, v \in S(u_0, r'). \quad (7.2)$$

Then an $\Omega = (u_0, r, a, b)$ -neighborhood of T_0 exists in which the equation

$$Tu = \theta, \quad (7.3)$$

is uniquely solvable and the solution $u(T)$ is continuous at $T = T_0$. More precisely in Ω we have.

$$\|u(T) - u_0\| \leq C \|Tu_0\| \quad \text{with a constant } C. \quad (7.4)$$

The easy proof follows immediately from Theorem 3.1 and supplement if we observe that, by (7.1),

$$T_0(u_0 + k) - Kk = Rk \quad \text{with} \quad Rk = o(\|k\|),$$

and, therefore, because of *b)* and *c)*, there exist positive numbers $r \leq r'$ and $m_1 < \|K^{-1}\|^{-1}$ with

$$\begin{aligned} \|K(u - v) - T_0u + T_0v\| &= \|T_0(u_0 + u - v) - T_0u + T_0v - R(u - v)\| \\ &\leq m_1 \|u - v\| \quad \text{for } u, v \in S(u_0, r). \end{aligned}$$

Supplement 7.1 a. Conditions *b)* and *c)* can be replaced by the following assumption:

b') At the point u_0 , T_0 has a strong derivative¹⁾ $T'_{0(u_0)} = K$ which has a bounded inverse, i.e. there exists a linear operator K with the property that to every $m > 0$ there is a $r > 0$ such that

$$\|T_0v - T_0u - K(v - u)\| \leq m \|v - u\| \quad \text{if } u, v \in S(u_0, r), \quad (7.5)$$

and K has a bounded inverse K^{-1} .

It is easy to show that *b')* implies *b)* and *c)* of Theorem 7.1 or directly $\alpha)$ and $\beta)$ of Theorem 3.1. Assumption *b')* again holds if we assume T_0 to have a derivative in a whole neighborhood of u_0 and this derivative is continuous and has a bounded inverse. But less is sufficient. More precisely we have the

Supplement 7.1 b. Condition *b')* holds if the following is true:

b'') T_0 has a (not necessarily bounded) derivative $T'_{0(u)}$ in a neighborhood $S(u_0, r)$ of u_0 with the property $T'_{0(u_0)} - T'_{0(u)}$ is bounded and $\|T'_{0(u_0)} - T'_{0(u)}\| \rightarrow 0$ as $\|u - u_0\| \rightarrow 0$ and $T'^{-1}_{0(u_0)}$ exists as a bounded operator.

The easy proof follows with $K = T'_{0(u_0)}$ from

$$\begin{aligned} \|T_0v - T_0u - K(v - u)\| &\leq \|T_0v - T_0u - T'_{0(u)}(v - u)\| \\ &\quad + \|T'_{0(u)} - T'_{0(u_0)}\| \|v - u\|. \end{aligned}$$

¹⁾ This notation is introduced by E. B. Leach [13] in connection with an inverse function theorem.

This supplement covers differential operators, for example, which usually are not continuous but have a continuous inverse. For such differential operators which have a derivative satisfying the assumptions $a)$ and $b')$ or $b'')$ the existence of an Ω -neighborhood can only fail at a "point" (T, u) where $T'_{(u)}{}^{-1}$ does not exist as a bounded linear operator. But the existence of a bounded inverse $T'_{(u)}{}^{-1}$ for each $u \in B_1$, T being defined everywhere in B_1 , is not sufficient to insure that T has an inverse nor that the equation $Tu = \omega$ is solvable for all $\omega \in B_2$.

8. ON THE DIFFERENTIABILITY OF THE SOLUTION.

In virtue of Theorem 7.1 and supplements the equation $Tu = \theta$ is equivalent to $u = u(T)$ in an Ω -neighborhood of (T_0, u_0) under the above conditions or, in other words, $u(T)$ is a unique function of T defined in Ω by $Tu = \theta$. The conditions yield also the continuity of $u(T)$ in the sense that $u(T)$ tends to u_0 as $\|Tu_0\| \rightarrow 0$ or, more precisely, $\|u(T) - u(T_0)\| \leq C \|Tu_0\|$ for some constant C . Therefore,

$$g(u) = o(\|u - u_0\|) \text{ implies } g(u) = o(\|Tu_0\|), \quad (8.1)$$

for these solutions $u = u(T)$ of $Tu = \theta$.

In order to get the continuity it is sufficient essentially that $\Delta T = T - T_0$ tends to zero at the single point u_0 . But for the purpose of calculating a Fréchet-derivative of $u(T)$ we have to know what the behaviour of T is in a neighborhood of u_0 as $\|Tu_0\| = \|\Delta Tu_0\| \rightarrow 0$. According to the definition of the derivative we are looking for a linear operator L such that the expression

$$u(T_0 + \Delta T) - u(T_0) - L\Delta T,$$

tends to zero faster than of order one as $\Delta T \rightarrow 0$ in a certain sense. But if we state the formula

$$\begin{aligned} u(T) - u(T_0) &= -T'_{0(u_0)}{}^{-1} \Delta Tu + o(\|u - u_0\|) \\ &= +T'_{0(u_0)}{}^{-1} T_0 u + o(\|u - u_0\|), \end{aligned} \quad (8.2)$$