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assertions remain true except the last one that  $T$  is a homeomorphism of  $B_1$  onto  $B_2$ . If there exist two subdomains  $D_a$  and  $D_a^*$  of  $D'$  then the assumptions of Theorem 6.1 cannot hold on a whole path  $P$  in  $B_1$  connecting  $D_a$  and  $D_a^*$ : Either  $T$  is not defined everywhere on  $P$  as a continuous operator or there does not exist an operator  $K$  with bounded inverse satisfying  $\alpha$ ),  $\beta$ ) and  $\gamma$ ) of Theorem 4.1.

A similar theorem can be stated using the assumptions of Theorem 4.1  $a$  as a basis.

## 7. DIFFERENTIABLE OPERATORS, IMPLICIT FUNCTION THEOREMS.

If the operator  $T$  is assumed to be differentiable in the sense of Fréchet (section 2  $c$ ) then the operator  $T'_{(u_0)}$  can be taken as operator  $K$  in the previous theorems and similar theorems can be stated.

THEOREM 7.1.  $a$ ) Let  $T_0$  be defined on the sphere  $S_0 = S(u_0, r_0) \subset B_1$  and let

$$T_0 u_0 = \theta. \quad (7.1)$$

$b$ ) Let  $T_0$  have a (not necessarily bounded) derivative  $T'_{0(u_0)} = K$  at the point  $u_0$  and let  $K$  have a bounded inverse  $K^{-1}$  defined on  $B_2$ .

$c$ ) Assume there are positive numbers  $r' \leq r_0$  and  $m = m(r') < \|K^{-1}\|^{-1}$  with

$$\|T_0(u_0 + u - v) - T_0 u + T_0 v\| \leq m \|u - v\|, \quad u, v \in S(u_0, r'). \quad (7.2)$$

Then an  $\Omega = (u_0, r, a, b)$ -neighborhood of  $T_0$  exists in which the equation

$$Tu = \theta, \quad (7.3)$$

is uniquely solvable and the solution  $u(T)$  is continuous at  $T = T_0$ . More precisely in  $\Omega$  we have.

$$\|u(T) - u_0\| \leq C \|Tu_0\| \quad \text{with a constant } C. \quad (7.4)$$

The easy proof follows immediately from Theorem 3.1 and supplement if we observe that, by (7.1),

$$T_0(u_0 + k) - Kk = Rk \quad \text{with} \quad Rk = o(\|k\|),$$

and, therefore, because of *b)* and *c)*, there exist positive numbers  $r \leq r'$  and  $m_1 < \|K^{-1}\|^{-1}$  with

$$\begin{aligned} \|K(u - v) - T_0u + T_0v\| &= \|T_0(u_0 + u - v) - T_0u + T_0v - R(u - v)\| \\ &\leq m_1 \|u - v\| \quad \text{for} \quad u, v \in S(u_0, r). \end{aligned}$$

*Supplement 7.1 a.* Conditions *b)* and *c)* can be replaced by the following assumption:

*b')* At the point  $u_0$ ,  $T_0$  has a strong derivative<sup>1)</sup>  $T'_{0(u_0)} = K$  which has a bounded inverse, i.e. there exists a linear operator  $K$  with the property that to every  $m > 0$  there is a  $r > 0$  such that

$$\|T_0v - T_0u - K(v - u)\| \leq m \|v - u\| \quad \text{if} \quad u, v \in S(u_0, r), \quad (7.5)$$

and  $K$  has a bounded inverse  $K^{-1}$ .

It is easy to show that *b')* implies *b)* and *c)* of Theorem 7.1 or directly  $\alpha)$  and  $\beta)$  of Theorem 3.1. Assumption *b')* again holds if we assume  $T_0$  to have a derivative in a whole neighborhood of  $u_0$  and this derivative is continuous and has a bounded inverse. But less is sufficient. More precisely we have the

*Supplement 7.1 b.* Condition *b')* holds if the following is true:

*b'')*  $T_0$  has a (not necessarily bounded) derivative  $T'_{0(u)}$  in a neighborhood  $S(u_0, r)$  of  $u_0$  with the property  $T'_{0(u_0)} - T'_{0(u)}$  is bounded and  $\|T'_{0(u_0)} - T'_{0(u)}\| \rightarrow 0$  as  $\|u - u_0\| \rightarrow 0$  and  $T'^{-1}_{0(u_0)}$  exists as a bounded operator.

The easy proof follows with  $K = T'_{0(u_0)}$  from

$$\begin{aligned} \|T_0v - T_0u - K(v - u)\| &\leq \|T_0v - T_0u - T'_{0(u)}(v - u)\| \\ &\quad + \|T'_{0(u)} - T'_{0(u_0)}\| \|v - u\|. \end{aligned}$$

<sup>1)</sup> This notation is introduced by E. B. Leach [13] in connection with an inverse function theorem.

This supplement covers differential operators, for example, which usually are not continuous but have a continuous inverse. For such differential operators which have a derivative satisfying the assumptions  $a)$  and  $b')$  or  $b'')$  the existence of an  $\Omega$ -neighborhood can only fail at a "point"  $(T, u)$  where  $T'_{(u)}^{-1}$  does not exist as a bounded linear operator. But the existence of a bounded inverse  $T'_{(u)}^{-1}$  for each  $u \in B_1$ ,  $T$  being defined everywhere in  $B_1$ , is not sufficient to insure that  $T$  has an inverse nor that the equation  $Tu = \omega$  is solvable for all  $\omega \in B_2$ .

## 8. ON THE DIFFERENTIABILITY OF THE SOLUTION.

In virtue of Theorem 7.1 and supplements the equation  $Tu = \theta$  is equivalent to  $u = u(T)$  in an  $\Omega$ -neighborhood of  $(T_0, u_0)$  under the above conditions or, in other words,  $u(T)$  is a unique function of  $T$  defined in  $\Omega$  by  $Tu = \theta$ . The conditions yield also the continuity of  $u(T)$  in the sense that  $u(T)$  tends to  $u_0$  as  $\|Tu_0\| \rightarrow 0$  or, more precisely,  $\|u(T) - u(T_0)\| \leq C \|Tu_0\|$  for some constant  $C$ . Therefore,

$$g(u) = o(\|u - u_0\|) \text{ implies } g(u) = o(\|Tu_0\|), \quad (8.1)$$

for these solutions  $u = u(T)$  of  $Tu = \theta$ .

In order to get the continuity it is sufficient essentially that  $\Delta T = T - T_0$  tends to zero at the single point  $u_0$ . But for the purpose of calculating a Fréchet-derivative of  $u(T)$  we have to know what the behaviour of  $T$  is in a neighborhood of  $u_0$  as  $\|Tu_0\| = \|\Delta Tu_0\| \rightarrow 0$ . According to the definition of the derivative we are looking for a linear operator  $L$  such that the expression

$$u(T_0 + \Delta T) - u(T_0) - L\Delta T,$$

tends to zero faster than of order one as  $\Delta T \rightarrow 0$  in a certain sense. But if we state the formula

$$\begin{aligned} u(T) - u(T_0) &= -T'_{0(u_0)} \Delta Tu + o(\|u - u_0\|) \\ &= +T'_{0(u_0)} T_0 u + o(\|u - u_0\|), \end{aligned} \quad (8.2)$$