

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 9 (1963)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS  
**Autor:** Ehrmann, Hans H.  
**Kapitel:** 7. DIFFERENTIABLE OPERATORS, IMPLICIT FUNCTION THEOREMS.  
**DOI:** <https://doi.org/10.5169/seals-38780>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 08.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

assertions remain true except the last one that  $T$  is a homeomorphism of  $B_1$  onto  $B_2$ . If there exist two subdomains  $D_a$  and  $D_a^*$  of  $D'$  then the assumptions of Theorem 6.1 cannot hold on a whole path  $P$  in  $B_1$  connecting  $D_a$  and  $D_a^*$ : Either  $T$  is not defined everywhere on  $P$  as a continuous operator or there does not exist an operator  $K$  with bounded inverse satisfying  $\alpha$ ),  $\beta$ ) and  $\gamma$ ) of Theorem 4.1.

A similar theorem can be stated using the assumptions of Theorem 4.1  $a$  as a basis.

## 7. DIFFERENTIABLE OPERATORS, IMPLICIT FUNCTION THEOREMS.

If the operator  $T$  is assumed to be differentiable in the sense of Fréchet (section 2  $c$ ) then the operator  $T'_{(u_0)}$  can be taken as operator  $K$  in the previous theorems and similar theorems can be stated.

THEOREM 7.1.  $a)$  Let  $T_0$  be defined on the sphere  $S_0 = S(u_0, r_0) \subset B_1$  and let

$$T_0 u_0 = \theta. \quad (7.1)$$

$b)$  Let  $T_0$  have a (not necessarily bounded) derivative  $T'_{0(u_0)} = K$  at the point  $u_0$  and let  $K$  have a bounded inverse  $K^{-1}$  defined on  $B_2$ .

$c)$  Assume there are positive numbers  $r' \leq r_0$  and  $m = m(r') < \|K^{-1}\|^{-1}$  with

$$\|T_0(u_0 + u - v) - T_0 u + T_0 v\| \leq m \|u - v\|, \quad u, v \in S(u_0, r'). \quad (7.2)$$

Then an  $\Omega = (u_0, r, a, b)$ -neighborhood of  $T_0$  exists in which the equation

$$Tu = \theta, \quad (7.3)$$

is uniquely solvable and the solution  $u(T)$  is continuous at  $T = T_0$ . More precisely in  $\Omega$  we have.

$$\|u(T) - u_0\| \leq C \|Tu_0\| \quad \text{with a constant } C. \quad (7.4)$$

The easy proof follows immediately from Theorem 3.1 and supplement if we observe that, by (7.1),

$$T_0(u_0 + k) - Kk = Rk \quad \text{with} \quad Rk = o(\|k\|),$$

and, therefore, because of *b)* and *c)*, there exist positive numbers  $r \leq r'$  and  $m_1 < \|K^{-1}\|^{-1}$  with

$$\begin{aligned} \|K(u - v) - T_0u + T_0v\| &= \|T_0(u_0 + u - v) - T_0u + T_0v - R(u - v)\| \\ &\leq m_1 \|u - v\| \quad \text{for} \quad u, v \in S(u_0, r). \end{aligned}$$

*Supplement 7.1 a.* Conditions *b)* and *c)* can be replaced by the following assumption:

*b')* At the point  $u_0$ ,  $T_0$  has a strong derivative<sup>1)</sup>  $T'_{0(u_0)} = K$  which has a bounded inverse, i.e. there exists a linear operator  $K$  with the property that to every  $m > 0$  there is a  $r > 0$  such that

$$\|T_0v - T_0u - K(v - u)\| \leq m \|v - u\| \quad \text{if} \quad u, v \in S(u_0, r), \quad (7.5)$$

and  $K$  has a bounded inverse  $K^{-1}$ .

It is easy to show that *b')* implies *b)* and *c)* of Theorem 7.1 or directly  $\alpha)$  and  $\beta)$  of Theorem 3.1. Assumption *b')* again holds if we assume  $T_0$  to have a derivative in a whole neighborhood of  $u_0$  and this derivative is continuous and has a bounded inverse. But less is sufficient. More precisely we have the

*Supplement 7.1 b.* Condition *b')* holds if the following is true:

*b'')*  $T_0$  has a (not necessarily bounded) derivative  $T'_{0(u)}$  in a neighborhood  $S(u_0, r)$  of  $u_0$  with the property  $T'_{0(u_0)} - T'_{0(u)}$  is bounded and  $\|T'_{0(u_0)} - T'_{0(u)}\| \rightarrow 0$  as  $\|u - u_0\| \rightarrow 0$  and  $T'^{-1}_{0(u_0)}$  exists as a bounded operator.

The easy proof follows with  $K = T'_{0(u_0)}$  from

$$\begin{aligned} \|T_0v - T_0u - K(v - u)\| &\leq \|T_0v - T_0u - T'_{0(u)}(v - u)\| \\ &\quad + \|T'_{0(u)} - T'_{0(u_0)}\| \|v - u\|. \end{aligned}$$

<sup>1)</sup> This notation is introduced by E. B. Leach [13] in connection with an inverse function theorem.

This supplement covers differential operators, for example, which usually are not continuous but have a continuous inverse. For such differential operators which have a derivative satisfying the assumptions  $a)$  and  $b')$  or  $b'')$  the existence of an  $\Omega$ -neighborhood can only fail at a "point"  $(T, u)$  where  $T'_{(u)}^{-1}$  does not exist as a bounded linear operator. But the existence of a bounded inverse  $T'_{(u)}^{-1}$  for each  $u \in B_1$ ,  $T$  being defined everywhere in  $B_1$ , is not sufficient to insure that  $T$  has an inverse nor that the equation  $Tu = \omega$  is solvable for all  $\omega \in B_2$ .

## 8. ON THE DIFFERENTIABILITY OF THE SOLUTION.

In virtue of Theorem 7.1 and supplements the equation  $Tu = \theta$  is equivalent to  $u = u(T)$  in an  $\Omega$ -neighborhood of  $(T_0, u_0)$  under the above conditions or, in other words,  $u(T)$  is a unique function of  $T$  defined in  $\Omega$  by  $Tu = \theta$ . The conditions yield also the continuity of  $u(T)$  in the sense that  $u(T)$  tends to  $u_0$  as  $\|Tu_0\| \rightarrow 0$  or, more precisely,  $\|u(T) - u(T_0)\| \leq C \|Tu_0\|$  for some constant  $C$ . Therefore,

$$g(u) = o(\|u - u_0\|) \text{ implies } g(u) = o(\|Tu_0\|), \quad (8.1)$$

for these solutions  $u = u(T)$  of  $Tu = \theta$ .

In order to get the continuity it is sufficient essentially that  $\Delta T = T - T_0$  tends to zero at the single point  $u_0$ . But for the purpose of calculating a Fréchet-derivative of  $u(T)$  we have to know what the behaviour of  $T$  is in a neighborhood of  $u_0$  as  $\|Tu_0\| = \|\Delta Tu_0\| \rightarrow 0$ . According to the definition of the derivative we are looking for a linear operator  $L$  such that the expression

$$u(T_0 + \Delta T) - u(T_0) - L\Delta T,$$

tends to zero faster than of order one as  $\Delta T \rightarrow 0$  in a certain sense. But if we state the formula

$$\begin{aligned} u(T) - u(T_0) &= -T'_{0(u_0)} \Delta Tu + o(\|u - u_0\|) \\ &= +T'_{0(u_0)} T_0 u + o(\|u - u_0\|), \end{aligned} \quad (8.2)$$