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Artikel: ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS

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assertions remain true except the last one that T is a homeomorphism of B_1 onto B_2. If there exist two subdomains D_a and $D_a{}^*$ of D' then the assumptions of Theorem 6.1 cannot hold on a whole path P in B_1 connecting D_a and $D_a{}^*$: Either T is not defined everywhere on P as a continuous operator or there does not exist an operator K with bounded inverse satisfying α), β) and γ) of Theorem 4.1.

A similar theorem can be stated using the assumptions of Theorem 4.1 *a* as a basis.

7. Differentiable operators, implicit function theorems.

If the operator T is assumed to be differentiable in the sense of Fréchet (section 2 *c*) then the operator $T'_{(u_0)}$ can be taken as operator K in the previous theorems and similar theorems can be stated.

Theorem 7.1. *a)* Let T_0 be defined on the sphere $S_0 = S(u_0, r_0) \subset B_1$ and let

$$T_0 u_0 = \theta . \tag{7.1}$$

b) Let T_0 have a (not necessarily bounded) derivative $T'_{0(u_0)} = K$ at the point u_0 and let K have a bounded inverse K^{-1} defined on B_2.

c) Assume there are positive numbers $r' \leqq r_0$ and $m = m(r') < \| K^{-1} \|^{-1}$ with

$$\| T_0(u_0 + u - v) - T_0 u + T_0 v \| \leqq m \| u - v \| , \; u , \; v \in S(u_0, r') . \tag{7.2}$$

Then an $\Omega = (u_0, r, a, b)$-neighborhood of T_0 exists in which the equation

$$Tu = \theta , \tag{7.3}$$

is uniquely solvable and the solution $u(T)$ is continuous at $T = T_0$. More precisely in Ω we have.

$$\| u(T) - u_0 \| \leqq C \| Tu_0 \| \quad \text{with a constant C.} \tag{7.4}$$

The easy proof follows immediately from Theorem 3.1 and supplement if we observe that, by (7.1),

$$T_0 (u_0 + k) - Kk = Rk \quad \text{with} \quad Rk = o(\| k \|),$$

and, therefore, because of *b)* and *c)*, there exist positive numbers $r \leqq r'$ and $m_1 < \| K^{-1} \|^{-1}$ with

$$\| K(u - v) - T_0 u + T_0 v \| = \| T_0(u_0 + u - v) - T_0 u + T_0 v - R(u - v) \|$$

$$\leqq m_1 \| u - v \| \quad \text{for} \quad u, v \in S(u_0, r).$$

Supplement 7.1 a. Conditions *b)* and *c)* can be replaced by the following assumption:

b') At the point u_0, T_0 has a strong derivative [1]) $T'_{0(u_0)} = K$ which has a bounded inverse, i.e. there exists a linear operator K with the property that to every $m > 0$ there is a $r > 0$ such that

$$\| T_0 v - T_0 u - K(v - u) \| \leqq m \| v - u \| \quad \text{if} \quad u, v \in S(u_0, r), \qquad (7.5)$$

and K has a bounded inverse K^{-1}.

It is easy to show that *b')* implies *b)* and *c)* of Theorem 7.1 or directly α) and β) of Theorem 3.1. Assumption *b')* again holds if we assume T_0 to have a derivative in a whole neighborhood of u_0 and this derivative is continuous and has a bounded inverse. But less is sufficient. More precisely we have the

Supplement 7.1 b. Condition *b')* holds if the following is true:

b'') T_0 has a (not necessarily bounded) derivative $T'_{0(u)}$ in a neighborhood $S(u_0, r)$ of u_0 with the property $T'_{0(u_0)} - T'_{0(u)}$ is bounded and $\| T'_{0(u_0)} - T'_{0(u)} \| \to 0$ as $\| u - u_0 \| \to 0$ and $T'^{-1}_{0(u_0)}$ exists as a bounded operator.

The easy proof follows with $K = T'_{0(u_0)}$ from

$$\| T_0 v - T_0 u - K(v - u) \| \leqq \| T_0 v - T_0 u - T'_{0(u)}(v - u) \|$$
$$+ \| T'_{0(u)} - T'_{0(u_0)} \| \; \| v - u \| .$$

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1) This notation is introduced by E. B. Leach [13] in connection with an inverse function theorem.

This supplement covers differential operators, for example, which usually are not continuous but have a continuous inverse. For such differential operators which have a derivative satisfying the assumptions *a)* and *b')* or *b'')* the existence of an Ω-neighborhood can only fail at a " point " (T, u) where $T'^{-1}_{(u)}$ does not exist as a bounded linear operator. But the existence of a bounded inverse $T'^{-1}_{(u)}$ for each $u \in B_1$, T being defined everywhere in B_1, is not sufficient to insure that T has an inverse nor that the equation $Tu = w$ is solvable for all $w \in B_2$.

8. On the differentiability of the solution.

In virtue of Theorem 7.1 and supplements the equation $Tu = \theta$ is equivalent to $u = u(T)$ in an Ω-neighborhood of (T_0, u_0) under the above conditions or, in other words, $u(T)$ is a unique function of T defined in Ω by $Tu = \theta$. The conditions yield also the continuity of $u(T)$ in the sense that $u(T)$ tends to u_0 as $\| Tu_0 \| \to 0$ or, more precisely, $\| u(T) - u(T_0) \| \leqq C \| Tu_0 \|$ for some constant C. Therefore,

$$g(u) = \mathrm{o}(\| u - u_0 \|) \quad \text{implies} \quad g(u) = \mathrm{o}(\| Tu_0 \|), \tag{8.1}$$

for these solutions $u = u(T)$ of $Tu = \theta$.

In order to get the continuity it is sufficient essentially that $\Delta T = T - T_0$ tends to zero at the single point u_0. But for the purpose of calculating a Fréchet-derivative of $u(T)$ we have to know what the behaviour of T is in a neighborhood of u_0 as $\| Tu_0 \| = \| \Delta T u_0 \| \to 0$. According to the definition of the derivative we are looking for a linear operator L such that the expression

$$u(T_0 + \Delta T) - u(T_0) - L \Delta T,$$

tends to zero faster than of order one as $\Delta T \to 0$ in a certain sense. But if we state the formula

$$\begin{aligned} u(T) - u(T_0) &= -T'^{-1}_{0(u_0)} \Delta T u + \mathrm{o}(\| u - u_0 \|) \\ &= +T'^{-1}_{0(u_0)} T_0 u + \mathrm{o}(\| u - u_0 \|), \end{aligned} \tag{8.2}$$