Then by the mean value theorem and because

$$\frac{d}{du}\frac{1}{\cos^2 u} = \frac{2 \sin u}{\cos^3 u},$$

is increasing for increasing

$$u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

it follows that

$$m(u) = \frac{1}{\cos^2(u+r)} - \frac{1}{\cos^2 u} \quad \text{for} \quad 0 \leqq u < \frac{\pi}{2} \quad \text{and} \quad u+r < \frac{\pi}{2}.$$

In the following we restrict ourselves to these u.

From the above we get

$$(\| K^{-1} \|^{-1} - m) r > \left(\frac{1}{\cos^2(u+r)} - \frac{4r}{\cos^3(u+r)}\right) r, \; 0 < r < \frac{\pi}{2} - u.$$

Now choosing r as the smallest positive solution of $r = r(u) = \frac{1}{8} \cos(u+r)$, which implies $u+r < \frac{\pi}{2}$, we get

$$(\| K^{-1} \|^{-1} - m) r > \frac{1}{16 \cos(u+r)} > \frac{1}{16}. \text{ [1]}$$

The same is true for $-\frac{\pi}{2} < u < 0$ as can be proved in the same way. Thus the conditions of Theorem 4.1 are valid. In particular γ) is true for $c = \frac{1}{16}$.

6. Inverse function theorems (continued).

As was indicated by the example $\tan u = w$ in the last chapter, the assumptions of the Theorems 4.1 and 4.1 a are not sufficient to insure that the operator T will have an inverse

[1] Here we use the fact that u is real.

defined on the whole space B_2, i.e. that the equation $Tu = w$ has exactly one solution for each w in B_2. We will now obtain conditions under which the existence of a local inverse implies the existence of a global inverse.

THEOREM 6.1. Let T satisfy the assumptions of Theorem 4.1 and let T be a continuous operator in its domain of definition, D.

Then there exists a finite or infinite number A of open connected domains $D_a \subset D$ with the properties:

$\bigcup_{a \in A} D_a = D$, for each $a \in A$ the restriction T_a of T on D_a is a homeomorphism[1]) of D_a onto B_2, and the sets D_a are mutually disjoint.

Furthermore, if T is defined on the whole Banach space B_1 then T is itself a homeomorphism of B_1 onto B_2.

This theorem implies that under the assumptions there is for each $w \in B_2$ the same finite or infinite number A of solutions of $Tu = w$, and each solution lies in a domain D_a for which the existence of a local inverse implies that of a global one.

Proof. *a)* We first prove the following statement: Let w_1 and w_2 be two points of B_2 with $\| w_1 - w_2 \| < c$ (c from γ) in Theorem 4.1) and let $Tu_1 = w_1$. The existence of at least one such u_1 follows from Theorem 4.1. Furthermore, it is shown that there exists a sphere $S(u_1, r_1) = S_1$ in which the equation $Tu = w$ has a unique solution $u(w)$ for all w with $\| w - w_1 \| < c$. Therefore there exists a unique solution u_2 in S_1 of $Tu = w_2$.

Conversely, let $S(u_2, r_2) = S_2$ the corresponding neighborhood of u_2 in which a unique solution $\tilde{u}$ of $Tu = \tilde{w}$ for $\| \tilde{w} - w_2 \| < c$ exists. Then $w = \tilde{w} \in S(w_1, c) \cap S(w_2, c)$, $u \in S(u_1, r_1)$, $\tilde{u} \in S(u_2, r_2)$, $Tu = w$, $T\tilde{u} = \tilde{w}$ implies $u = \tilde{u}$. If $u \in S_2$ the assertion is true because of the uniqueness of $\tilde{u} = u(\tilde{w})$ in S_2 for $\| \tilde{w} - w_2 \| < c$. Now, let $u \notin S_2$. Then we connect w_2 with w by the straight line $g = w_2 + \lambda (w - w_2)$, $0 \leqq \lambda \leqq 1$, and consider the images C_1 and C_2 of this line in S_1 and S_2, respectively. These images exist and form connected curves $\varphi_i(\lambda) \in S_i$, $i = 1, 2$, using the fact that

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[1]) One-to-one mapping continuous and with continuous inverse.

$g \in S(w_1, c) \cap S(w_2, c)$ in B_2 and applying the theorem that the continuous image of a connected set is connected, which holds in our spaces. We also have $\varphi_i(0) = u_2$, $i = 1, 2$, $\varphi_1(1) = u$, $\varphi_2(1) = \tilde{u}$. In the intersection $S_1 \cap S_2$ the curves C_i coincide because of the uniqueness of $u(w)$, $\tilde{u}(w)$ in S_1, S_2 respectively.

We proceed with increasing λ from u_2 along C_1. Since $u \notin S_2$ there is a first point u^* (with a least $\lambda = \lambda^*$) on C_1 which does not belong to $C_2 \in S_2$. However, in each neighborhood of u^* there are points of C_2. Let $w^* = w_2 + \lambda^*(w - w_2)$, the corresponding point with $Tu^* = w^*$. Then, because of the continuity of C_2, there cannot be another point u on C_2 with $Tu = w^*$, i.e. $u^* \in S_2$ and $C_1 = C_2$ in contradiction to our assumption.

b) Let u_0 be a solution of $Tu = \theta$, which exists by Theorem 4. This theorem also yields a neighborhood $S(u_0, r_0) = S_0$ such that the equation $Tu = w$ has a unique solution $u(w)$ in S_0 for all w with $\| w \| \leqq c - \epsilon$, $0 < \epsilon < c$, and $u(w)$ is continuous there.

We choose a number $R > 0$ arbitrarily large and construct a continuous mapping T_a^{-1} with $T_a^{-1} T = I$ defined for all w with $\| w \| \leqq R$ and with range in a certain domain of B_1. This can be done as follows:

For $\| w \| \leqq c - \epsilon$ the equation $Tu = w$ has a unique and continuous solution, $u(w)$, if u is prescribed to lie in S_0. The (inverse-) images u for these w form a connected closed set in B_1. Let $Tu = w$ be uniquely solvable for all w in the disk $\| w \| \leqq R_1$ by the continuous function $u = u(w)$ and let the set $D_{(R_1)} = \{ u = u(w) : \| w \| \leqq R_1 \}$ be a connected, closed set containing the point u_0.

Because of the continuity of T the restriction of T to $D_{(R_1)}$ is a one-to-one mapping of $D_{(R_1)}$ onto $\bar{S}(\theta, R_1) \subset B_2$ which is continuous in both directions, i.e. a homeomorphism. In particular, the intersection $S(\tilde{w}, c) \cap \bar{S}(\theta, R_1)$ has its preimage in the corresponding intersection $S(\tilde{u}, r) \cap D_{(R_1)}$ for each $\tilde{w} \in \bar{S}(\theta, R_1)$ with $T\tilde{u} = \tilde{w}$.

Now we consider the sphere $\| w \| \leqq R_1 + \frac{c}{2} = R_2$. Each w in the shell $R_1 < \| w \| \leqq R_2$ lies in some sphere $\| w - \tilde{w} \| < c$ with $\| \tilde{w} \| \leqq R_1$. We assign to these w the $u = u(w)$ with $Tu = w$ which lies in the corresponding neighborhood $S(\tilde{u}, \tilde{r})$ with $T\tilde{u} = \tilde{w}$. This defines $u(w)$ uniquely. This follows from *a)* since if $\tilde{w}_1$ and $\tilde{w}_2$ are two points in $S(\theta, R_1)$ with $\| w - w_i \| < c$, $i = 1, 2$, then w, w_1 and w_2 lie also in the sphere $S(w^*, c)$ with $w^* = \frac{1}{2}(w_1 + w_2)$ and $\| w^* \| \leqq R_1$. Therefore, it follows from *a)* that our assumptions stated for $\| w \| \leqq R_1$ are true also for $\| w \| \leqq R_1 + \frac{c}{2}$.

Thus, we get a homeomorphism between a certain domain $D_a \subset B_1$ and B_2. Contrary to the case of a linear operator there may be more than one such domain. If there is another solution $u^* \notin D_a$ of $Tu = w^*$ for any $w^* \in B_2$ then by the same construction, with w^* as new center, we obtain another domain $D_a{}^*$, and the restriction of T to $D_a{}^*$ is a homeomorphism on $D_a{}^*$ onto B_2.

We prove that D_a and $D_a{}^*$ are disjoint. Let $\tilde{u} \in D_a \cap D_a{}^*$. Then we connect $\tilde{u}$ with u^* by a curve C^* lying in $D_a{}^*$. This curve has an image TC^* in B_2, which is also a curve because of the continuity of T. TC^* has an inverse image $C_a' = T_a^{-1}\, TC^*$ in D_a given by the homeomorphism D_a onto B_2, which is also a curve. C_a' and C^* coincide in $D_a \cap D_a{}^*$. Let u' be the first point of C^* from $\tilde{u}$ lying on the boundary of D_a. This exists since $u^* \notin D_a$. Then it follows from the continuity of C_a' that $u' \in C_a' \subset D_a$, in contradiction to the openess of D_a. Therefore, D_a and $D_a{}^*$ are disjoint.

Let T be defined on the whole space B_1. If there is only one domain D_a then the assertion is true. Let there be at least two such domains. Then by a similar consideration connecting two points, $u \in D_a$ and $u^* \in D_a{}^*$, with the same image by a curve one finds that T cannot be defined on the boundary of such a domain D_a. This contradicts the assumption and completes the proof.

Corollary. If we merely require the assumptions of Theorem 6.1 to be satisfied on a subdomain $D' \subset D$ then all assertions remain true except the last one that T is a homeomorphism of B_1 onto B_2. If there exist two subdomains D_a and $D_a{}^*$ of D' then the assumptions of Theorem 6.1 cannot hold on a whole path P in B_1 connecting D_a and $D_a{}^*$: Either T is not defined everywhere on P as a continuous operator or there does not exist an operator K with bounded inverse satisfying α), β) and γ) of Theorem 4.1.

A similar theorem can be stated using the assumptions of Theorem 4.1 *a* as a basis.

7. Differentiable operators, implicit function theorems.

If the operator T is assumed to be differentiable in the sense of Fréchet (section 2 *c*) then the operator $T'_{(u_0)}$ can be taken as operator K in the previous theorems and similar theorems can be stated.

Theorem 7.1. *a)* Let T_0 be defined on the sphere $S_0 = S(u_0, r_0) \subset B_1$ and let

$$T_0 u_0 = \theta . \tag{7.1}$$

b) Let T_0 have a (not necessarily bounded) derivative $T'_{0(u_0)} = K$ at the point u_0 and let K have a bounded inverse K^{-1} defined on B_2.

c) Assume there are positive numbers $r' \leqq r_0$ and $m = m(r') < \| K^{-1} \|^{-1}$ with

$$\| T_0(u_0 + u - v) - T_0 u + T_0 v \| \leqq m \| u - v \| , \; u , \; v \in S(u_0, r') . \tag{7.2}$$

Then an $\Omega = (u_0, r, a, b)$-neighborhood of T_0 exists in which the equation

$$Tu = \theta , \tag{7.3}$$

is uniquely solvable and the solution $u(T)$ is continuous at $T = T_0$. More precisely in Ω we have.

$$\| u(T) - u_0 \| \leqq C \| Tu_0 \| \quad \text{with a constant C.} \tag{7.4}$$