## 6. Inverse function theorems (continued).

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Then by the mean value theorem and because

$$
\frac{d}{d u} \frac{1}{\cos ^{2} u}=\frac{2 \sin u}{\cos ^{3} u}
$$

is increasing for increasing

$$
u \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)
$$

it follows that
$m(u)=\frac{1}{\cos ^{2}(u+r)}-\frac{1}{\cos ^{2} u} \quad$ for $\quad 0 \leqq u<\frac{\pi}{2} \quad$ and $\quad u+r<\frac{\pi}{2}$.
In the following we restrict ourselves to these $u$.
From the above we get
$\left(\left\|K^{-1}\right\|^{-1}-m\right) r>\left(\frac{1}{\cos ^{2}(u+r)}-\frac{4 r}{\cos ^{3}(u+r)}\right) r, 0<r<\frac{\pi}{2}-u$.
Now choosing $r$ as the smallest positive solution of $r=r(u)=\frac{1}{8} \cos (u+r)$, which implies $u+r<\frac{\pi}{2}$, we get

$$
\left.\left(\left\|K^{-1}\right\|^{-1}-m\right) r>\frac{1}{16 \cos (u+r)}>\frac{1}{16} \cdot{ }^{1}\right)
$$

The same is true for $-\frac{\pi}{2}<u<0$ as can be proved in the same way. Thus the conditions of Theorem 4.1 are valid. In particular $\gamma$ ) is true for $c=\frac{1}{16}$.
6. Inverse function theorems (continued).

As was indicated by the example $\tan u=w$ in the last chapter, the assumptions of the Theorems 4.1 and $4.1 a$ are not sufficient to insure that the operator $T$ will have an inverse

[^0]defined on the whole space $B_{2}$, i.e. that the equation $T u=w$ has exactly one solution for each $w$ in $B_{2}$. We will now obtain conditions under which the existence of a local inverse implies the existence of a global inverse.

Theorem 6.1. Let $T$ satisfy the assumptions of Theorem 4.1 and let $T$ be a continuous operator in its domain of definition, $D$.

Then there exists a finite or infinite number $A$ of open connected domains $D_{a} \subset D$ with the properties:
$\underset{a \in A}{U} D_{a}=D$, for each $a \in A$ the restriction $T_{a}$ of $T$ on $D_{a}$ is a homeomorphism ${ }^{1}$ ) of $D_{a}$ onto $B_{2}$, and the sets $D_{a}$ are mutually disjoint.

Furthermore, if $T$ is defined on the whole Banach space $B_{1}$ then $T$ is itself a homeomorphism of $B_{1}$ onto $B_{2}$.

This theorem implies that under the assumptions there is for each $\mathfrak{x \in B _ { 2 }}$ the same finite or infinite number $A$ of solutions of $T u=w$, and each solution lies in a domain $D_{a}$ for which the existence of a local inverse implies that of a global one.

Proof. a) We first prove the following statement: Let $w_{1}$ and $w_{2}$ be two points of $B_{2}$ with $\left\|w_{1}-w_{2}\right\|<c(c$ from $\gamma)$ in Theorem 4.1) and let $T u_{1}=w_{1}$. The existence of at least one such $u_{1}$ follows from Theorem 4.1. Furthermore, it is shown that there exists a sphere $S\left(u_{1}, r_{1}\right)=S_{1}$ in which the equation $T u=w$ has a unique solution $u(w)$ for all $w$ with $\left\|w-w_{1}\right\|<c$. Therefore there exists a unique solution $u_{2}$ in $S_{1}$ of $T u=\mathscr{w}_{2}$.

Conversely, let $S\left(u_{2}, r_{2}\right)=S_{2}$ the corresponding neighborhood of $u_{2}$ in which a unique solution $\tilde{u}$ of $T u=\tilde{w}$ for $\left\|\tilde{w}-\mathscr{w}_{2}\right\|<c \quad$ exists. Then $w=\tilde{w} \in S\left(w_{1}, c\right) \cap S\left(\mathscr{w}_{2}, c\right)$, $u \in S\left(u_{1}, r_{1}\right), \quad \tilde{u} \in S\left(u_{2}, r_{2}\right), \quad T u=w, T \tilde{u}=\tilde{w}$ implies $u=\tilde{u}$. If $u \in S_{2}$ the assertion is true because of the uniqueness of $\tilde{u}=u(\tilde{w})$ in $S_{2}$ for $\left\|\tilde{w}-\mathfrak{W}_{2}\right\|<c$. Now, let $u \notin S_{2}$. Then we connect $W_{2}$ with $\mathscr{W}$ by the straight line $g=\mathscr{W}_{2}+\lambda\left(\mathscr{W}-\mathscr{W}_{2}\right)$, $0 \leqq \lambda \leqq 1$, and consider the images $C_{1}$ and $C_{2}$ of this line in $S_{1}$ and $S_{2}$, respectively. These images exist and form connected curves $\varphi_{i}(\lambda) \in S_{i}, i=1,2$, using the fact that

[^1]$g \in S\left(w_{1}, c\right) \cap S\left(w_{2}, c\right)$ in $B_{2}$ and applying the theorem that the continuous image of a connected set is connected, which holds in our spaces. We also have $\varphi_{i}(0)=u_{2}, i=1,2$, $\varphi_{1}(1)=u, \quad \varphi_{2}(1)=\tilde{u} . \quad$ In the intersection $S_{1} \cap S_{2}$ the curves $C_{i}$ coincide because of the uniqueness of $u(w), \tilde{u}(w)$ in $S_{1}, S_{2}$ respectively.

We proceed with increasing $\lambda$ from $u_{2}$ along $C_{1}$. Since $u \notin S_{2}$ there is a first point $u^{*}$ (with a least $\lambda=\lambda^{*}$ ) on $C_{1}$ which does not belong to $C_{2} \in S_{2}$. However, in each neighborhood of $u^{*}$ there are points of $C_{2}$. Let $w^{*}=W_{2}+\lambda^{*}\left(w-w_{2}\right)$, the corresponding point with $T u^{*}=w^{*}$. Then, because of the continuity of $C_{2}$, there cannot be another point $u$ on $C_{2}$ with $T u=w^{*}$, i.e. $u^{*} \in S_{2}$ and $C_{1}=C_{2}$ in contradiction to our assumption.
b) Let $u_{0}$ be a solution of $T u=\theta$, which exists by Theorem 4 . This theorem also yields a neighborhood $S\left(u_{0}, r_{0}\right)=S_{0}$ such that the equation $T u=w$ has a unique solution $u(w)$ in $S_{0}$ for all $w$ with $\|w\| \leqq c-\epsilon, 0<\epsilon<c$, and $u(w)$ is continuous there.

We choose a number $R>0$ arbitrarily large and construct a continuous mapping $T_{a}{ }^{-1}$ with $T_{a}{ }^{-1} T=I$ defined for all $w$ with $\|\propto\| \leqq R$ and with range in a certain domain of $B_{1}$. This can be done as follows:

For $\|w\| \leqq c-\epsilon$ the equation $T u=w$ has a unique and continuous solution, $u(w)$, if $u$ is prescribed to lie in $S_{0}$. The (inverse-) images $u$ for these $w$ form a connected closed set in $B_{1}$. Let $T u=w$ be uniquely solvable for all $w$ in the disk $\|w\| \leqq R_{1}$ by the continuous function $u=u(w)$ and let the set $D_{\left(R_{1}\right)}=\left\{u=u(w):\|w\| \leqq R_{1}\right\}$ be a connected, closed. set containing the point $u_{0}$.

Because of the continuity of $T$ the restriction of $T$ to $D_{\left(R_{1}\right)}$ is a one-to-one mapping of $D_{\left(R_{1}\right)}$ onto $\bar{S}\left(\theta, R_{1}\right) \subset B_{2}$ which is continuous in both directions, i.e. a homeomorphism. In particular, the intersection $S(\tilde{w}, c) \cap \bar{S}\left(\theta, R_{1}\right)$ has its preimage in the corresponding intersection $S(\tilde{u}, r) \cap D_{\left(R_{1}\right)}$ for each $\tilde{w} \in \bar{S}\left(\theta, R_{1}\right)$ with $T \tilde{u}=\tilde{w}$.

Now we consider the sphere $\|w\| \leqq R_{1}+\frac{c}{2}=R_{2}$. Each $w$ in the shell $R_{1}<\|w\| \leqq R_{2}$ lies in some sphere $\|w-\tilde{w}\|<c$ with $\|\tilde{w}\| \leqq R_{1}$. We assign to these $w$ the $u=u(w)$ with $T u=\mathscr{w}$ which lies in the corresponding neighborhood $S(\tilde{u}, \tilde{r})$ with $T \tilde{u}=\tilde{w}$. This defines $u(w)$ uniquely. This follows from $a)$ since if $\tilde{w}_{1}$ and $\tilde{w}_{2}$ are two points in $S\left(\theta, R_{1}\right)$ with $\left\|\nsim-w_{i}\right\|<c, i=1,2$, then $w, w_{1}$ and $\mathscr{w}_{2}$ lie also in the sphere $S\left(w_{w^{*}}, c\right)$ with $w^{*}=\frac{1}{2}\left(w_{1}+w_{2}\right)$ and $\left\|w^{*}\right\| \leqq R_{1}$. Therefore, it follows from $a$ ) that our assumptions stated for $\|\mathscr{W}\| \leqq R_{1}$ are true also for $\|\rightsquigarrow\| \leqq R_{1}+\frac{c}{2}$.

Thus, we get a homeomorphism between a certain domain $D_{a} \subset B_{1}$ and $B_{2}$. Contrary to the case of a linear operator there may be more than one such domain. If there is another solution $u^{*} \notin D_{a}$ of $T u=w^{*}$ for any $w^{*} \in B_{2}$ then by the same construction, with $w^{*}$ as new center, we obtain another domain $D_{a}{ }^{*}$, and the restriction of $T$ to $D_{a}{ }^{*}$ is a homeomorphism on $D_{a}{ }^{*}$ onto $B_{2}$.

We prove that $D_{a}$ and $D_{a}{ }^{*}$ are disjoint. Let $\tilde{u} \in D_{a} \cap D_{a}{ }^{*}$. Then we connect $\tilde{u}$ with $u^{*}$ by a curve $C^{*}$ lying in $D_{a}^{*}$. This curve has an image $T C^{*}$ in $B_{2}$, which is also a curve because of the continuity of $T . T C^{*}$ has an inverse image $C_{a}^{\prime}=T_{a}^{-1} T C^{*}$ in $D_{a}$ given by the homeomorphism $D_{a}$ onto $B_{2}$, which is also a curve. $C_{a}^{\prime}$ and $C^{*}$ coincide in $D_{a} \cap D_{a}^{*}$. Let $u^{\prime}$ be the first point of $C^{*}$ from $\tilde{u}$ lying on the boundary of $D_{a}$. This exists since $u^{*} \notin D_{a}$. Then it follows from the continuity of $C_{a}^{\prime}$ that $u^{\prime} \in C_{a}^{\prime} \subset D_{a}$, in contradiction to the openess of $D_{a}$. Therefore, $D_{a}$ and $D_{a}^{*}$ are disjoint.

Let $T$ be defined on the whole space $B_{1}$. If there is only one domain $D_{a}$ then the assertion is true. Let there be at least two such domains. Then by a similar consideration connecting two points, $u \in D_{a}$ and $u^{*} \in D_{a}{ }^{*}$, with the same image by a curve one finds that $T$ cannot be defined on the boundary of such a domain $D_{a}$. This contradicts the assumption and completes the proof.

Corollary: If we merely require the assumptions of Theorem 6.1 to be satisfied on a subdomain $D^{\prime} \subset D$ then all
assertions remain true except the last one that $T$ is a homeomorphism of $B_{1}$ onto $B_{2}$. If there exist two subdomains $D_{a}$ and $D_{a}^{*}$ of $D^{\prime}$ then the assumptions of Theorem 6.1 cannot hold on a whole path $P$ in $B_{1}$ connecting $D_{a}$ and $D_{a}^{*}$ : Either $T$ is not defined everywhere on $P$ as a continuous operator or there does not exist an operator $K$ with bounded inverse satisfying $\alpha$ ), $\beta$ ) and $\gamma$ ) of Theorem 4.1.

A similar theorem can be stated using the assumptions of Theorem $4.1 a$ as a basis.

## 7. Differentiable operators, implicit function theorems.

If the operator $T$ is assumed to be differentiable in the sense of Fréchet (section $2 c$ ) then the operator $T_{\left(u_{0}\right)}^{\prime}$ can be taken as operator $K$ in the previous theorems and similar theorems can be stated.

Theorem 7.1. a) Let $T_{0}$ be defined on the sphere $S_{0}=S\left(u_{0}, r_{0}\right) \subset B_{1}$ and let

$$
\begin{equation*}
T_{0} u_{0}=\theta \tag{7.1}
\end{equation*}
$$

b) Let $T_{0}$ have a (not necessarily bounded) derivative $T_{0\left(u_{0}\right)}^{\prime}=K$ at the point $u_{0}$ and let $K$ have a bounded inverse $K^{-1}$ defined on $B_{2}$.
c) Assume there are positive numbers $r^{\prime} \leqq r_{0}$ and $m=m\left(r^{\prime}\right)<\left\|K^{-1}\right\|^{-1}$ with
$\left\|T_{0}\left(u_{0}+u-v\right)-T_{0} u+T_{0} v\right\| \leqq m\|u-v\|, u, v \in S\left(u_{0}, r^{\prime}\right)$.
Then an $\Omega=\left(u_{0}, r, a, b\right)$-neighborhood of $T_{0}$ exists in which the equation

$$
\begin{equation*}
T u=\theta, \tag{7.3}
\end{equation*}
$$

is uniquely solvable and the solution $u(T)$ is continuous at $T=T_{0}$. More precisely in $\Omega$ we have.

$$
\begin{equation*}
\left\|u(T)-u_{0}\right\| \leqq C\left\|T u_{0}\right\| \quad \text { with a constant C. } \tag{7.4}
\end{equation*}
$$


[^0]:    1) Here we use the fact that $u$ is real.
[^1]:    1) One-to-one mapping continuous and with continuous inverse.
