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Then by the mean value theorem and because

$$\frac{d}{du} \frac{1}{\cos^2 u} = \frac{2 \sin u}{\cos^3 u},$$

is increasing for increasing

$$u \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right),$$

it follows that

$$m(u) = \frac{1}{\cos^2(u+r)} - \frac{1}{\cos^2 u} \quad \text{for} \quad 0 \leq u < \frac{\pi}{2} \quad \text{and} \quad u+r < \frac{\pi}{2}.$$

In the following we restrict ourselves to these u .

From the above we get

$$(\|K^{-1}\|^{-1} - m)r > \left(\frac{1}{\cos^2(u+r)} - \frac{4r}{\cos^3(u+r)} \right) r, \quad 0 < r < \frac{\pi}{2} - u.$$

Now choosing r as the smallest positive solution of $r = r(u) = \frac{1}{8} \cos(u+r)$, which implies $u+r < \frac{\pi}{2}$, we get

$$(\|K^{-1}\|^{-1} - m)r > \frac{1}{16 \cos(u+r)} > \frac{1}{16}.^{1)}$$

The same is true for $-\frac{\pi}{2} < u < 0$ as can be proved in the same way. Thus the conditions of Theorem 4.1 are valid. In particular $\gamma)$ is true for $c = \frac{1}{16}$.

6. INVERSE FUNCTION THEOREMS (continued).

As was indicated by the example $\tan u = \omega$ in the last chapter, the assumptions of the Theorems 4.1 and 4.1 α are not sufficient to insure that the operator T will have an inverse

¹⁾ Here we use the fact that u is real.

defined on the whole space B_2 , i.e. that the equation $Tu = \omega$ has exactly one solution for each ω in B_2 . We will now obtain conditions under which the existence of a local inverse implies the existence of a global inverse.

THEOREM 6.1. Let T satisfy the assumptions of Theorem 4.1 and let T be a continuous operator in its domain of definition, D .

Then there exists a finite or infinite number A of open connected domains $D_a \subset D$ with the properties:

$\bigcup_{a \in A} D_a = D$, for each $a \in A$ the restriction T_a of T on D_a is a homeomorphism¹⁾ of D_a onto B_2 , and the sets D_a are mutually disjoint.

Furthermore, if T is defined on the whole Banach space B_1 then T is itself a homeomorphism of B_1 onto B_2 .

This theorem implies that under the assumptions there is for each $\omega \in B_2$ the same finite or infinite number A of solutions of $Tu = \omega$, and each solution lies in a domain D_a for which the existence of a local inverse implies that of a global one.

Proof. a) We first prove the following statement: Let ω_1 and ω_2 be two points of B_2 with $\|\omega_1 - \omega_2\| < c$ (c from γ) in Theorem 4.1) and let $Tu_1 = \omega_1$. The existence of at least one such u_1 follows from Theorem 4.1. Furthermore, it is shown that there exists a sphere $S(u_1, r_1) = S_1$ in which the equation $Tu = \omega$ has a unique solution $u(\omega)$ for all ω with $\|\omega - \omega_1\| < c$. Therefore there exists a unique solution u_2 in S_1 of $Tu = \omega_2$.

Conversely, let $S(u_2, r_2) = S_2$ the corresponding neighborhood of u_2 in which a unique solution \tilde{u} of $Tu = \tilde{\omega}$ for $\|\tilde{\omega} - \omega_2\| < c$ exists. Then $\omega = \tilde{\omega} \in S(\omega_1, c) \cap S(\omega_2, c)$, $u \in S(u_1, r_1)$, $\tilde{u} \in S(u_2, r_2)$, $Tu = \omega$, $T\tilde{u} = \tilde{\omega}$ implies $u = \tilde{u}$. If $u \in S_2$ the assertion is true because of the uniqueness of $\tilde{u} = u(\tilde{\omega})$ in S_2 for $\|\tilde{\omega} - \omega_2\| < c$. Now, let $u \notin S_2$. Then we connect ω_2 with ω by the straight line $g = \omega_2 + \lambda(\omega - \omega_2)$, $0 \leq \lambda \leq 1$, and consider the images C_1 and C_2 of this line in S_1 and S_2 , respectively. These images exist and form connected curves $\varphi_i(\lambda) \in S_i$, $i = 1, 2$, using the fact that

1) One-to-one mapping continuous and with continuous inverse.

$g \in S(\omega_1, c) \cap S(\omega_2, c)$ in B_2 and applying the theorem that the continuous image of a connected set is connected, which holds in our spaces. We also have $\varphi_i(0) = u_2$, $i = 1, 2$, $\varphi_1(1) = u$, $\varphi_2(1) = \tilde{u}$. In the intersection $S_1 \cap S_2$ the curves C_i coincide because of the uniqueness of $u(\omega)$, $\tilde{u}(\omega)$ in S_1 , S_2 respectively.

We proceed with increasing λ from u_2 along C_1 . Since $u \notin S_2$ there is a first point u^* (with a least $\lambda = \lambda^*$) on C_1 which does not belong to $C_2 \in S_2$. However, in each neighborhood of u^* there are points of C_2 . Let $\omega^* = \omega_2 + \lambda^*(\omega - \omega_2)$, the corresponding point with $Tu^* = \omega^*$. Then, because of the continuity of C_2 , there cannot be another point u on C_2 with $Tu = \omega^*$, i.e. $u^* \in S_2$ and $C_1 = C_2$ in contradiction to our assumption.

b) Let u_0 be a solution of $Tu = \theta$, which exists by Theorem 4. This theorem also yields a neighborhood $S(u_0, r_0) = S_0$ such that the equation $Tu = \omega$ has a unique solution $u(\omega)$ in S_0 for all ω with $\|\omega\| \leq c - \epsilon$, $0 < \epsilon < c$, and $u(\omega)$ is continuous there.

We choose a number $R > 0$ arbitrarily large and construct a continuous mapping T_a^{-1} with $T_a^{-1}T = I$ defined for all ω with $\|\omega\| \leq R$ and with range in a certain domain of B_1 . This can be done as follows:

For $\|\omega\| \leq c - \epsilon$ the equation $Tu = \omega$ has a unique and continuous solution, $u(\omega)$, if u is prescribed to lie in S_0 . The (inverse-) images u for these ω form a connected closed set in B_1 . Let $Tu = \omega$ be uniquely solvable for all ω in the disk $\|\omega\| \leq R_1$ by the continuous function $u = u(\omega)$ and let the set $D_{(R_1)} = \{u = u(\omega) : \|\omega\| \leq R_1\}$ be a connected, closed set containing the point u_0 .

Because of the continuity of T the restriction of T to $D_{(R_1)}$ is a one-to-one mapping of $D_{(R_1)}$ onto $\bar{S}(\theta, R_1) \subset B_2$ which is continuous in both directions, i.e. a homeomorphism. In particular, the intersection $S(\tilde{\omega}, c) \cap \bar{S}(\theta, R_1)$ has its pre-image in the corresponding intersection $S(\tilde{u}, r) \cap D_{(R_1)}$ for each $\tilde{\omega} \in \bar{S}(\theta, R_1)$ with $T\tilde{u} = \tilde{\omega}$.

Now we consider the sphere $\|\omega\| \leq R_1 + \frac{c}{2} = R_2$. Each ω in the shell $R_1 < \|\omega\| \leq R_2$ lies in some sphere $\|\omega - \tilde{\omega}\| < c$ with $\|\tilde{\omega}\| \leq R_1$. We assign to these ω the $u = u(\omega)$ with $Tu = \omega$ which lies in the corresponding neighborhood $S(\tilde{u}, \tilde{r})$ with $T\tilde{u} = \tilde{\omega}$. This defines $u(\omega)$ uniquely. This follows from a) since if $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are two points in $S(\theta, R_1)$ with $\|\omega - \omega_i\| < c$, $i = 1, 2$, then ω , ω_1 and ω_2 lie also in the sphere $S(\omega^*, c)$ with $\omega^* = \frac{1}{2}(\omega_1 + \omega_2)$ and $\|\omega^*\| \leq R_1$. Therefore, it follows from a) that our assumptions stated for $\|\omega\| \leq R_1$ are true also for $\|\omega\| \leq R_1 + \frac{c}{2}$.

Thus, we get a homeomorphism between a certain domain $D_a \subset B_1$ and B_2 . Contrary to the case of a linear operator there may be more than one such domain. If there is another solution $u^* \notin D_a$ of $Tu = \omega^*$ for any $\omega^* \in B_2$ then by the same construction, with ω^* as new center, we obtain another domain D_a^* , and the restriction of T to D_a^* is a homeomorphism on D_a^* onto B_2 .

We prove that D_a and D_a^* are disjoint. Let $\tilde{u} \in D_a \cap D_a^*$. Then we connect \tilde{u} with u^* by a curve C^* lying in D_a^* . This curve has an image TC^* in B_2 , which is also a curve because of the continuity of T . TC^* has an inverse image $C'_a = T_a^{-1} TC^*$ in D_a given by the homeomorphism D_a onto B_2 , which is also a curve. C'_a and C^* coincide in $D_a \cap D_a^*$. Let u' be the first point of C^* from \tilde{u} lying on the boundary of D_a . This exists since $u^* \notin D_a$. Then it follows from the continuity of C'_a that $u' \in C'_a \subset D_a$, in contradiction to the openness of D_a . Therefore, D_a and D_a^* are disjoint.

Let T be defined on the whole space B_1 . If there is only one domain D_a then the assertion is true. Let there be at least two such domains. Then by a similar consideration connecting two points, $u \in D_a$ and $u^* \in D_a^*$, with the same image by a curve one finds that T cannot be defined on the boundary of such a domain D_a . This contradicts the assumption and completes the proof.

Corollary. If we merely require the assumptions of Theorem 6.1 to be satisfied on a subdomain $D' \subset D$ then all

assertions remain true except the last one that T is a homeomorphism of B_1 onto B_2 . If there exist two subdomains D_a and D_a^* of D' then the assumptions of Theorem 6.1 cannot hold on a whole path P in B_1 connecting D_a and D_a^* : Either T is not defined everywhere on P as a continuous operator or there does not exist an operator K with bounded inverse satisfying α), β) and γ) of Theorem 4.1.

A similar theorem can be stated using the assumptions of Theorem 4.1 a as a basis.

7. DIFFERENTIABLE OPERATORS, IMPLICIT FUNCTION THEOREMS.

If the operator T is assumed to be differentiable in the sense of Fréchet (section 2 c) then the operator $T'_{(u_0)}$ can be taken as operator K in the previous theorems and similar theorems can be stated.

THEOREM 7.1. $a)$ Let T_0 be defined on the sphere $S_0 = S(u_0, r_0) \subset B_1$ and let

$$T_0 u_0 = \theta. \quad (7.1)$$

$b)$ Let T_0 have a (not necessarily bounded) derivative $T'_{0(u_0)} = K$ at the point u_0 and let K have a bounded inverse K^{-1} defined on B_2 .

$c)$ Assume there are positive numbers $r' \leq r_0$ and $m = m(r') < \|K^{-1}\|^{-1}$ with

$$\|T_0(u_0 + u - v) - T_0 u + T_0 v\| \leq m \|u - v\|, \quad u, v \in S(u_0, r'). \quad (7.2)$$

Then an $\Omega = (u_0, r, a, b)$ -neighborhood of T_0 exists in which the equation

$$Tu = \theta, \quad (7.3)$$

is uniquely solvable and the solution $u(T)$ is continuous at $T = T_0$. More precisely in Ω we have.

$$\|u(T) - u_0\| \leq C \|Tu_0\| \quad \text{with a constant } C. \quad (7.4)$$