Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 9 (1963)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS
Autor: Ehrmann, Hans H.
Kapitel: 4. Inverse function theorems.
DOI: https://doi.org/10.5169/seals-38780

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 20.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Then, for the solution $u = u(T)$ in S, the estimate (3.7) holds.

A unique solution of (3.1 a) in $S(u^*, r)$ also exists for such r and b if (3.6) holds, but in (3.8) the sign " $>$ " cannot be replaced by " $\geqq$ ", nor can the constant a in (3.8) be replaced by any larger one.

The last statement can be proved by simple examples in the one-dimensional case and with an operator T which is linear in $S(u^*, r)$.

4. Inverse function theorems.

Under the conditions of the implicit function Theorem 3.1, the operator T has a local inverse defined in a neighborhood of a point w_0 for which

$$Tu = w. \tag{4.1}$$

has a solution u_0. This inverse has its range in a neighborhood of u_0. For the proof set $T^*u = Tu - w_0$ in Theorem 3.1. However, the conditions of this theorem are still not sufficient for the existence of a solution u of equation (4.1) for all w in B_2 even if T is defined on the whole Banach space B_1 and the conditions are satisfied at each point u of B_1.[1])

However, this actually is not necessary for the existence of at least one solution u of (4.1) for all $w \in B_2$ as is indicated by the following theorem.

Theorem 4.1. Let the operator T, mapping a non-empty domain $D \subset B_1$ into B_2, satisfy the following conditions:

For each $u \in D$ there exist a sphere $S(u, r) \subset D$, a linear operator K, and a constant m such that the following conditions hold:

α) K has a bounded inverse K^{-1} on $TS(u, r)$

β) $\| Tv - T\tilde{v} - K(v - \tilde{v}) \| \leqq m \| v - \tilde{v} \|$ for $v, \tilde{v} \in S(u, r)$

γ) $(\| K^{-1} \|^{-1} - m)\, r \geqq c > 0$ where the constant c is independent of $u \in D$.

---

1) Example: Tu ≡ arctan u = w, with $B_1 = B_2 =$ { real numbers }, is not solvable for all w∈ B_2, although the conditions of Theorem 3.1 are satisfied at each point (u, w = arctan u) for T*u = Tu-w.

Then the equation (4.1) has at least one solution for every w in B_2 and each $w_0 \in B_2$ is the center of a sphere $\| w - w_0 \| < c$ for which $u = u(w)$ is continuous and unique in a corresponding neighborhood $S(u_0, r_0)$ with $Tu_0 = w_0$ and $r_0 = r(u_0)$.

Remark. In this theorem it is not required that T be defined for all $u \in B_1$ nor that T be continuous, and it cannot be asserted that T has only one solution for each $w \in B_2$. The example in the footnote (previous page) shows that the condition $\gamma)$ cannot be improved by deleting $\geqq c$ with constant c independent of u. But $\gamma)$ can be replaced by other conditions.

THEOREM 4.1 *a*. In Theorem 4.1 the condition $\gamma)$ can be replaced by

$\gamma')$ There exists for each $R > 0$ a constant $c = c(R) > 0$ such that

$$(\| K^{-1} \|^{-1} - m) r \geqq c \quad \text{for} \quad \| u \| \leqq R, \text{ and} \tag{4.2}$$

$$\| Tu \| \to \infty \quad \text{as} \quad \| u \| \to \infty \quad \text{and} \quad u \in D. \tag{4.3}$$

Proof of Theorem 4.1. a) Let $u_0 \in D$ and $Tu_0 = w_0$ and let K_0, m_0, r_0 be the corresponding quantities satisfying $\alpha)$, $\beta)$ and $\gamma)$ with $S_0 = S(u_0, r_0) \subset D$.
Then by Theorem 3.1 and supplement with $T^*u = Tu - w_0$, $u^* = u_0$, $r = r^* = r_0$ and $b = 0$, it follows that each equation $\tilde{T}u = Tu - w = \theta$ has a unique solution $u(w)$ in S_0 which depends continuously on w provided

$$\| K_0^{-1} \tilde{T} u_0 \| < (1 - m_0 \| K_0^{-1} \|) r_0 = \| K_0^{-1} \| (\| K_0^{-1} \|^{-1} - m_0) r_0 .$$

Because of

$$\| K_0^{-1} \tilde{T} u_0 \| \leqq \| K_0^{-1} \| \cdot \| w_0 - w \|,$$

and $\gamma)$ this inequality holds for $\| w - w_0 \| < c$, i.e. (4.1) has a solution $u(w)$ for these w. The solution $u(w)$ is unique and continuous in S_0.

b) Let w_1 be an arbitrary point in B_2. Then the non-empty set Λ of all real λ with $0 \leqq \lambda \leqq 1$ for which the equation

$$Tu - w_0 + \lambda (w_0 - w_1) = \theta,$$

is solvable is open with respect to the interval [0, 1]. This follows from *a)*. It is also closed, for if $\tilde{\lambda}$ is the supremum of Λ then there exists a point $\lambda^* \in \Lambda$ with $|\tilde{\lambda}^* - \lambda| \, \| w_0 - w_1 \| < c$. Thus it follows from *a)*, if w_0 is replaced by $w_0 - \lambda^* (w_0 - w_1)$, that $\tilde{\lambda} \in \Lambda$. Hence $\Lambda = [0, 1]$ and (4.1) has a solution for all $w \in B_2$.

Proof of Theorem 4.1 a. Let $w_1 \in B_2$ and $u_0 \in D$ with $Tu_0 = w_0$ be given. Then the points $w = w_0 + \lambda (w_1 - w_0)$, $0 \leqq \lambda \leqq 1$, are bounded:

$$\| w \| \leqq \max (\| w_0 \|, \| w_1 \|) = A .$$

Because of $\gamma')$ there exists a number R with $\| Tu \| > A$ for all u in the set $\{ u \in D : \| u \| \geqq R \}$.[1])

Then the same conclusion as in the proof of Theorem 4.1 with $c = c(R)$ applied to $\| u \| \leqq R$ shows that $Tu = w_1$ is solvable by an element u_1 with $\| u_1 \| < R$ for which the assumptions of Theorem 4.1 with $c = c(R)$ hold. This implies the existence of a sphere $\| w - w_1 \| < c$ with the asserted properties.

5. An example.

The simple example $Tu = \tan u$, given only for illustration purposes, shows that Theorem 4.1 is general enough to cover cases in which either the domain D is not the whole space B_1 or $Tu = w$ does not have a unique solution, although this equation is solvable for all $w \in B_2$.

Let $B_1 = B_2 = B$ be the Banach space of real numbers. Then by Theorem 4.1 the equation

$$Tu \equiv \tan u = w , \quad u, w \in B ,$$

is solvable for all $w \in B$.[2])

Proof. We choose

$$Kv = \frac{v}{\cos^2 u} \quad \text{for} \quad u \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

---

1) This set may be empty.

2) This is not true for complex numbers as tan u = i is not solvable.