4. Inverse function theorems.

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Then, for the solution $u = u(T)$ in S, the estimate (3.7) holds.

A unique solution of (3.1 a) in $S(u^*, r)$ also exists for such r and b if (3.6) holds, but in (3.8) the sign " $>$ " cannot be replaced by " $\geqq$ ", nor can the constant a in (3.8) be replaced by any larger one.

The last statement can be proved by simple examples in the one-dimensional case and with an operator T which is linear in $S(u^*, r)$.

4. Inverse function theorems.

Under the conditions of the implicit function Theorem 3.1, the operator T has a local inverse defined in a neighborhood of a point w_0 for which

$$Tu = w. \tag{4.1}$$

has a solution u_0. This inverse has its range in a neighborhood of u_0. For the proof set $T^*u = Tu - w_0$ in Theorem 3.1. However, the conditions of this theorem are still not sufficient for the existence of a solution u of equation (4.1) for all w in B_2 even if T is defined on the whole Banach space B_1 and the conditions are satisfied at each point u of B_1.[1])

However, this actually is not necessary for the existence of at least one solution u of (4.1) for all $w \in B_2$ as is indicated by the following theorem.

Theorem 4.1. Let the operator T, mapping a non-empty domain $D \subset B_1$ into B_2, satisfy the following conditions:

For each $u \in D$ there exist a sphere $S(u, r) \subset D$, a linear operator K, and a constant m such that the following conditions hold:

α) K has a bounded inverse K^{-1} on $TS(u, r)$

β) $\| Tv - T\tilde{v} - K(v - \tilde{v}) \| \leqq m \| v - \tilde{v} \|$ for $v, \tilde{v} \in S(u, r)$

γ) $(\| K^{-1} \|^{-1} - m) r \geqq c > 0$ where the constant c is independent of $u \in D$.

1) Example: $Tu \equiv \arctan u = w$, with $B_1 = B_2 = \{$real numbers$\}$, is not solvable for all $w \in B_2$, although the conditions of Theorem 3.1 are satisfied at each point $(u, w = \arctan u)$ for $T^*u = Tu - w$.

Then the equation (4.1) has at least one solution for every w in B_2 and each $w_0 \in B_2$ is the center of a sphere $\| w - w_0 \| < c$ for which $u = u(w)$ is continuous and unique in a corresponding neighborhood $S(u_0, r_0)$ with $Tu_0 = w_0$ and $r_0 = r(u_0)$.

Remark. In this theorem it is not required that T be defined for all $u \in B_1$ nor that T be continuous, and it cannot be asserted that T has only one solution for each $w \in B_2$. The example in the footnote (previous page) shows that the condition γ) cannot be improved by deleting $\geqq c$ with constant c independent of u. But γ) can be replaced by other conditions.

THEOREM 4.1 *a.* In Theorem 4.1 the condition γ) can be replaced by

γ') There exists for each $R > 0$ a constant $c = c(R) > 0$ such that

$$(\| K^{-1} \|^{-1} - m) r \geqq c \quad \text{for} \quad \| u \| \leqq R, \text{ and} \tag{4.2}$$

$$\| Tu \| \to \infty \quad \text{as} \quad \| u \| \to \infty \quad \text{and} \quad u \in D. \tag{4.3}$$

Proof of Theorem 4.1. a) Let $u_0 \in D$ and $Tu_0 = w_0$ and let K_0, m_0, r_0 be the corresponding quantities satisfying α), β) and γ) with $S_0 = S(u_0, r_0) \subset D$.
Then by Theorem 3.1 and supplement with $T^*u = Tu - w_0$, $u^* = u_0$, $r = r^* = r_0$ and $b = 0$, it follows that each equation $\tilde{T}u = Tu - w = \theta$ has a unique solution $u(w)$ in S_0 which depends continuously on w provided

$$\| K_0^{-1} \tilde{T}u_0 \| < (1 - m_0 \| K_0^{-1} \|) r_0 = \| K_0^{-1} \| (\| K_0^{-1} \|^{-1} - m_0) r_0 .$$

Because of

$$\| K_0^{-1} \tilde{T}u_0 \| \leqq \| K_0^{-1} \| \cdot \| w_0 - w \| ,$$

and γ) this inequality holds for $\| w - w_0 \| < c$, i.e. (4.1) has a solution $u(w)$ for these w. The solution $u(w)$ is unique and continuous in S_0.

b) Let w_1 be an arbitrary point in B_2. Then the non-empty set Λ of all real λ with $0 \leqq \lambda \leqq 1$ for which the equation

$$Tu - w_0 + \lambda (w_0 - w_1) = \theta ,$$

is solvable is open with respect to the interval [0, 1]. This follows from *a)*. It is also closed, for if $\tilde{\lambda}$ is the supremum of Λ then there exists a point $\lambda^* \in \Lambda$ with $| \tilde{\lambda}^* - \lambda | \, \| w_0 - w_1 \| < c$. Thus it follows from *a)*, if w_0 is replaced by $w_0 - \lambda^* (w_0 - w_1)$, that $\tilde{\lambda} \in \Lambda$. Hence $\Lambda = [0, 1]$ and (4.1) has a solution for all $w \in B_2$.

Proof of Theorem 4.1 a. Let $w_1 \in B_2$ and $u_0 \in D$ with $Tu_0 = w_0$ be given. Then the points $w = w_0 + \lambda (w_1 - w_0)$, $0 \leqq \lambda \leqq 1$, are bounded:

$$\| w \| \leqq \max (\| w_0 \|, \| w_1 \|) = A .$$

Because of γ') there exists a number R with $\| Tu \| > A$ for all u in the set $\{ u \in D : \| u \| \geqq R \}$.[1])

Then the same conclusion as in the proof of Theorem 4.1 with $c = c(R)$ applied to $\| u \| \leqq R$ shows that $Tu = w_1$ is solvable by an element u_1 with $\| u_1 \| < R$ for which the assumptions of Theorem 4.1 with $c = c(R)$ hold. This implies the existence of a sphere $\| w - w_1 \| < c$ with the asserted properties.

5. An example.

The simple example $Tu = \tan u$, given only for illustration purposes, shows that Theorem 4.1 is general enough to cover cases in which either the domain D is not the whole space B_1 or $Tu = w$ does not have a unique solution, although this equation is solvable for all $w \in B_2$.

Let $B_1 = B_2 = B$ be the Banach space of real numbers. Then by Theorem 4.1 the equation

$$Tu \equiv \tan u = w , \quad u, w \in B ,$$

is solvable for all $w \in B$.[2])

Proof. We choose

$$Kv = \frac{v}{\cos^2 u} \quad \text{for} \quad u \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

1) This set may be empty.
2) This is not true for complex numbers as tan u = i is not solvable.