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Then, for the solution $u = u(T)$ in S , the estimate (3.7) holds.

A unique solution of (3.4 a) in $S(u^*, r)$ also exists for such r and b if (3.6) holds, but in (3.8) the sign " $>$ " cannot be replaced by " \geq ", nor can the constant a in (3.8) be replaced by any larger one.

The last statement can be proved by simple examples in the one-dimensional case and with an operator T which is linear in $S(u^*, r)$.

4. INVERSE FUNCTION THEOREMS.

Under the conditions of the implicit function Theorem 3.1, the operator T has a local inverse defined in a neighborhood of a point w_0 for which

$$Tu = w. \quad (4.1)$$

has a solution u_0 . This inverse has its range in a neighborhood of u_0 . For the proof set $T^*u = Tu - w_0$ in Theorem 3.1. However, the conditions of this theorem are still not sufficient for the existence of a solution u of equation (4.1) for all w in B_2 even if T is defined on the whole Banach space B_1 and the conditions are satisfied at each point u of B_1 .¹⁾

However, this actually is not necessary for the existence of at least one solution u of (4.1) for all $w \in B_2$ as is indicated by the following theorem.

THEOREM 4.1. Let the operator T , mapping a non-empty domain $D \subset B_1$ into B_2 , satisfy the following conditions:

For each $u \in D$ there exist a sphere $S(u, r) \subset D$, a linear operator K , and a constant m such that the following conditions hold:

- $\alpha)$ K has a bounded inverse K^{-1} on $TS(u, r)$
- $\beta)$ $\|T\varphi - T\tilde{\varphi} - K(\varphi - \tilde{\varphi})\| \leq m \|\varphi - \tilde{\varphi}\|$ for $\varphi, \tilde{\varphi} \in S(u, r)$
- $\gamma)$ $(\|K^{-1}\|^{-1} - m)r \geq c > 0$ where the constant c is independent of $u \in D$.

1) Example: $Tu \equiv \arctan u = w$, with $B_1 = B_2 = \{\text{real numbers}\}$, is not solvable for all $w \in B_2$, although the conditions of Theorem 3.1 are satisfied at each point $(u, w = \arctan u)$ for $T^*u = Tu - w$.

Then the equation (4.1) has at least one solution for every ω in B_2 and each $\omega_0 \in B_2$ is the center of a sphere $\|\omega - \omega_0\| < c$ for which $u = u(\omega)$ is continuous and unique in a corresponding neighborhood $S(u_0, r_0)$ with $Tu_0 = \omega_0$ and $r_0 = r(u_0)$.

Remark. In this theorem it is not required that T be defined for all $u \in B_1$ nor that T be continuous, and it cannot be asserted that T has only one solution for each $\omega \in B_2$. The example in the footnote (previous page) shows that the condition $\gamma)$ cannot be improved by deleting $\geq c$ with constant c independent of u . But $\gamma)$ can be replaced by other conditions.

THEOREM 4.1 a. In Theorem 4.1 the condition $\gamma)$ can be replaced by

$\gamma')$ There exists for each $R > 0$ a constant $c = c(R) > 0$ such that

$$(\|K^{-1}\|^{-1} - m)r \geq c \quad \text{for} \quad \|u\| \leq R, \quad \text{and} \quad (4.2)$$

$$\|Tu\| \rightarrow \infty \quad \text{as} \quad \|u\| \rightarrow \infty \quad \text{and} \quad u \in D. \quad (4.3)$$

Proof of Theorem 4.1. a) Let $u_0 \in D$ and $Tu_0 = \omega_0$ and let K_0, m_0, r_0 be the corresponding quantities satisfying $\alpha)$, $\beta)$ and $\gamma)$ with $S_0 = S(u_0, r_0) \subset D$.

Then by Theorem 3.1 and supplement with $T^*u = Tu - \omega_0$, $u^* = u_0$, $r = r^* = r_0$ and $b = 0$, it follows that each equation $\tilde{T}u = Tu - \omega = \theta$ has a unique solution $u(\omega)$ in S_0 which depends continuously on ω provided

$$\|K_0^{-1} \tilde{T}u_0\| < (1 - m_0 \|K_0^{-1}\|)r_0 = \|K_0^{-1}\|(\|K_0^{-1}\|^{-1} - m_0)r_0.$$

Because of

$$\|K_0^{-1} \tilde{T}u_0\| \leq \|K_0^{-1}\| \cdot \|\omega_0 - \omega\|,$$

and $\gamma)$ this inequality holds for $\|\omega - \omega_0\| < c$, i.e. (4.1) has a solution $u(\omega)$ for these ω . The solution $u(\omega)$ is unique and continuous in S_0 .

b) Let ω_1 be an arbitrary point in B_2 . Then the non-empty set A of all real λ with $0 \leq \lambda \leq 1$ for which the equation

$$Tu - \omega_0 + \lambda(\omega_0 - \omega_1) = \theta,$$

is solvable is open with respect to the interval $[0, 1]$. This follows from $a)$. It is also closed, for if $\tilde{\lambda}$ is the supremum of A then there exists a point $\lambda^* \in A$ with $|\tilde{\lambda}^* - \lambda| \|\omega_0 - \omega_1\| < c$. Thus it follows from $a)$, if ω_0 is replaced by $\omega_0 - \lambda^* (\omega_0 - \omega_1)$, that $\tilde{\lambda} \in A$. Hence $A = [0, 1]$ and (4.1) has a solution for all $\omega \in B_2$.

Proof of Theorem 4.1 a. Let $\omega_1 \in B_2$ and $u_0 \in D$ with $Tu_0 = \omega_0$ be given. Then the points $\omega = \omega_0 + \lambda (\omega_1 - \omega_0)$, $0 \leq \lambda \leq 1$, are bounded:

$$\|\omega\| \leq \max(\|\omega_0\|, \|\omega_1\|) = A.$$

Because of $\gamma')$ there exists a number R with $\|Tu\| > A$ for all u in the set $\{u \in D: \|u\| \geq R\}$.¹⁾

Then the same conclusion as in the proof of Theorem 4.1 with $c = c(R)$ applied to $\|u\| \leq R$ shows that $Tu = \omega_1$ is solvable by an element u_1 with $\|u_1\| < R$ for which the assumptions of Theorem 4.1 with $c = c(R)$ hold. This implies the existence of a sphere $\|\omega - \omega_1\| < c$ with the asserted properties.

5. AN EXAMPLE.

The simple example $Tu = \tan u$, given only for illustration purposes, shows that Theorem 4.1 is general enough to cover cases in which either the domain D is not the whole space B_1 or $Tu = \omega$ does not have a unique solution, although this equation is solvable for all $\omega \in B_2$.

Let $B_1 = B_2 = B$ be the Banach space of real numbers. Then by Theorem 4.1 the equation

$$Tu \equiv \tan u = w, \quad u, w \in B,$$

is solvable for all $\omega \in B$.²⁾

Proof. We choose

$$Kv = \frac{v}{\cos^2 u} \quad \text{for} \quad u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

¹⁾ This set may be empty.

²⁾ This is not true for complex numbers as $\tan u = i$ is not solvable.