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Artikel: ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS

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shell $R \leqq \| u - u_0 \| \leqq R_1$, the thickness of which is the smallest possible.

The last section has a quite different character from the previous ones. It contains as an essential result a theorem for the unique solvability of a certain linear equation involving a completely continuous symmetric linear operator. These investigations are of a strictly linear kind, using the theory of eigenvalues of such operators, but applied to special non-linear equations as, for example, non-linear differentiable integral equations of Hammerstein type. They enable us to give explicit conditions on the derivative in order to insure the existence of a solution. This generalizes known existence theorems for such equations.

The literature in the field which is treated here is so extensive that it is impossible to mention all related works.

2. Notations and preliminaries.

a) Throughout this paper the letters B_i, $i = 1, 2, \ldots$, denote Banach spaces with norms $\| u \|_i$, $u \in B_i$, and zero-elements θ_i. For the sake of simplicity we omit the indices on the norms and zero-elements if there is no danger of confusion.

The empty set is denoted by $\emptyset$.

$S(u^*, r)$, means an open, and $\bar{S}(u^*, r)$ *a* closed spherical neighborhood with center u^* and radius r, i.e., the sets

$$\{u : \| u - u^* \| < r\}, \quad \text{respectively} \quad \{u : \| u - u^* \| \leqq r\}.$$

b) We are dealing with (in general non-linear) operators T, V, ... defined on (open) domains D, D_V, ... of Banach spaces and with ranges R, R_V, ... in Banach spaces. We write $T \in (D \to B)$ if $TD = R \subset B$. We assume throughout this paper that the arguments of the operators always lie in the domains of definition if no confusion can occur. The operator I denotes the identity mapping.

For $$\frac{\| Th \|}{\| Gh \|} \to 0, \text{ or } \frac{\| Th \|}{\| Gh \|} \leqq C \text{ as } h \to \theta, \ h \in D,$$

we write equivalently

$$Th = o(\| Gh \|), \quad \text{or} \quad Th = O(\| Gh \|).$$

T is continuous at $u \in D$ if $T(u+h) - Tu = o(1)$.

c) By a Fréchet-differential (F-differential) of an operator T at a point $u \in D \subset B$ we understand an expression $T'_{(u)} k$ with a (not necessarily bounded [1]) linear operator $T'_{(u)}$ defined on B for which

$$T(u+k) - Tu - T'_{(u)} k = R(u, k) = o(\| k \|).$$

$T'_{(u)}$ is called the Fréchet-derivative (F-derivative).

Let T have a bounded F-derivative for all u of the straight line $u = u_0 + tk$, $0 \leqq t \leqq 1$, then the mean value theorem [2]) holds:

$$\| T(u+k) - Tu \| \leqq \sup_{0 \leqq t \leqq 1} \| T'_{(u+tk)} \| \cdot \| k \|.$$

If T is continuous and differentiable at u, then $T'_{(u)}$ is a continuous operator. This follows from

$$\| T'_{(u)} k \| \leqq \| T(u+k) - Tu - T'_{(u)} k \| + \| T(u+k) - Tu \| \\ = o(1) \quad \text{for} \quad k \to \theta.$$

If $T'_{(u)} k$ has a F-differential with respect to u, i.e.

$$T'_{(u+k_2)} k_1 - T'_{(u)} k_1 - T''_{(u)} k_1 k_2 = o(\| k_2 \|),$$

the operator $T''_{(u)}$ is called the second F-derivative of T. $T''_{(u)}$ is a bilinear operator operating on k_1 and k_2.

d) The operator T is called completely continuous [3]) or compact if it maps each bounded set of its domain $D \subset B_1$ in a conditionally compact subset S of its range $R \subset B_2$, that is, in a set $S \subset R$ each infinite sequence of which contains a subsequence which converges to some element of B_2.

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1) For applications it is sometimes more convenient to admit unbounded operators here.

2) See, for example, L. V. Kantorovich [2], p. 162.

3) See ,for example, E. Hille and R. S. Phillips [4], p. 48, or A. E. Taylor [5], p. 274 .

For compact operators the Schauder fixed point theorem [1]) holds:

Let the compact operator T map the convex, closed set $M \subset D$ into M: $TM \subset M$. Then there exists a fixed point u^* of T in M, that is, a point $u^* = Tu^*$.

e) In the following we often consider equations of the form

$$Tu \equiv (T_0 + \Delta T)u = \theta\,, \quad T,\ T_0,\ \Delta T \in (D \to B_2)\,, \qquad (2.1)$$

with an operator T which lies in a certain neighborhood of T_0 with respect to a sphere $S(u_0, r)$ of its domain. For the purpose of formulating some neighborhood theorems for those operators we introduce the notation of a (u_0, r, a, b)-neighborhood, also called an Ω-neighborhood, of an operator T_0 with respect to $S(u_0, r)$:

Definition. T is said to be lying in an $\Omega = (u_0, r, a, b)$-neighborhood of the operator T_0 if and only if

$$\| (T - T_0)u_0 \| = \| \Delta T u_0 \| < a\,, \qquad (2.2a)$$

$$\| \Delta Tu - \Delta Tv \| \leqq b \| u - v \| \text{ for all } u, v \in S(u_0, r)\,, \qquad (2.2b)$$

where $\Delta T = T - T_0$ and $S(u_0, r) \subset D_T \cap D_{T_0}$.

If T has these properties we briefly write $T \in \Omega$.

f) For some proofs we apply the contraction mapping theorem in the following well known form:

Theorem of contraction mappings [2]). Let V be a contracting operator which maps a closed region $S \subset B_1$ into itself, i.e.

$$\| Vu - Vv \| \leqq l \| u - v \|\,, \ l < 1\,, \quad \text{for} \quad u, v \in S\,, \qquad (2.3)$$

and

$$VS \subset S\,. \qquad (2.4)$$

Then in S, V has exactly one fixed point, $u = Vu$.

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[1]) J. Schauder [6], for a generalization see A. Tychonoff [7].

[2]) See, for example, J. Weissinger [8] who gave a more general form of this theorem. Without the estimate (2.6), the theorem was used by T. H. Hildebrandt and L. M. Graves [1], p. 133, for the proof of implicit function theorems. Nowadays it is basic for many error estimates in numerical analysis, see L. Collatz [9], p. 36ff. For generalizations see, for example, L. Kantorovich [2], [3], J. Schröder [11], H. Ehrmann [12].

Condition (2.4) is satisfied if (2.3) holds in the sphere

$$S: \| u - u_0 \| \leqq (1-l)^{-1} \| Vu_0 - u_0 \| . \qquad (2.5)$$

Moreover, u is the limit of the sequence $\{ u_n \}$ where

$$u_{u+1} = Vu_n , \qquad n = 0, 1, 2, \ldots ,$$

and there results the estimate

$$\| u - u_{n+1} \| \leqq l(1-l)^{-1} \| u_{n+1} - u_n \| \leqq l^{n+1} (1-l)^{-1} \| u_1 - u_0 \| . \qquad (2.6)$$

3. The implicit function theorem.

Theorem 3.1. Let T^* be an operator with domain $D \subset B_1$ and range in B_2, let $S^* = S(u^*, r^*) \subset D$ and

$$T^* u^* = \theta . \qquad (3.1)$$

We assume furthermore that there exists a linear operator K on S^* into B_2 with the following properties:

α) K has a bounded inverse, K^{-1}, defined on B_2 and
β) There exists a constant $m < \| K^{-1} \|^{-1}$ such that

$$\| T^* v - T^* u - K(v-u) \| \leqq m \| v - u \| \quad \text{for} \quad u, v \in S^* . \qquad (3.2)$$

Then there exists an $\Omega = (u^*, r, a, b)$-neighborhood of T^*, such that for all $T \in \Omega$ the equation

$$Tu = \theta , \qquad (3.1a)$$

has a unique solution $u = u(T)$ in $S(u^*, r)$. This solution is continuous in T at $T = T^*$ in the sense

$$\| u(T) - u^* \| \to 0 \quad \text{as} \quad \| Tu^* \| \to 0 . \qquad (3.3)$$

In this theorem the operators T and K need not be continuous.