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shell  $R \leq \|u - u_0\| \leq R_1$ , the thickness of which is the smallest possible.

The last section has a quite different character from the previous ones. It contains as an essential result a theorem for the unique solvability of a certain linear equation involving a completely continuous symmetric linear operator. These investigations are of a strictly linear kind, using the theory of eigenvalues of such operators, but applied to special non-linear equations as, for example, non-linear differentiable integral equations of Hammerstein type. They enable us to give explicit conditions on the derivative in order to insure the existence of a solution. This generalizes known existence theorems for such equations.

The literature in the field which is treated here is so extensive that it is impossible to mention all related works.

## 2. NOTATIONS AND PRELIMINARIES.

a) Throughout this paper the letters  $B_i$ ,  $i = 1, 2, \dots$ , denote Banach spaces with norms  $\|u\|_i$ ,  $u \in B_i$ , and zero-elements  $\theta_i$ . For the sake of simplicity we omit the indices on the norms and zero-elements if there is no danger of confusion.

The empty set is denoted by  $\emptyset$ .

$S(u^*, r)$ , means an open, and  $\bar{S}(u^*, r)$  a closed spherical neighborhood with center  $u^*$  and radius  $r$ , i.e., the sets

$$\{u: \|u - u^*\| < r\}, \quad \text{respectively} \quad \{u: \|u - u^*\| \leq r\}.$$

b) We are dealing with (in general non-linear) operators  $T, V, \dots$  defined on (open) domains  $D, D_V, \dots$  of Banach spaces and with ranges  $R, R_V, \dots$  in Banach spaces. We write  $T \in (D \rightarrow B)$  if  $TD = R \subset B$ . We assume throughout this paper that the arguments of the operators always lie in the domains of definition if no confusion can occur. The operator  $I$  denotes the identity mapping.

$$\text{For} \quad \frac{\|Th\|}{\|Gh\|} \rightarrow 0, \quad \text{or} \quad \frac{\|Th\|}{\|Gh\|} \leq C \quad \text{as } h \rightarrow \theta, \quad h \in D,$$

we write equivalently

$$Th = o(\| Gh \|), \quad \text{or} \quad Th = O(\| Gh \|).$$

$T$  is continuous at  $u \in D$  if  $T(u+h) - Tu = o(1)$ .

c) By a Fréchet-differential ( $F$ -differential) of an operator  $T$  at a point  $u \in D \subset B$  we understand an expression  $T'_{(u)} k$  with a (not necessarily bounded<sup>1)</sup> linear operator  $T'_{(u)}$  defined on  $B$  for which

$$T(u+k) - Tu - T'_{(u)} k = R(u, k) = o(\| k \|).$$

$T'_{(u)}$  is called the Fréchet-derivative ( $F$ -derivative).

Let  $T$  have a bounded  $F$ -derivative for all  $u$  of the straight line  $u = u_0 + tk$ ,  $0 \leq t \leq 1$ , then the mean value theorem<sup>2)</sup> holds:

$$\| T(u+k) - Tu \| \leq \sup_{0 \leq t \leq 1} \| T'_{(u+tk)} \| \cdot \| k \|.$$

If  $T$  is continuous and differentiable at  $u$ , then  $T'_{(u)}$  is a continuous operator. This follows from

$$\begin{aligned} \| T'_{(u)} k \| &\leq \| T(u+k) - Tu - T'_{(u)} k \| + \| T(u+k) - Tu \| \\ &= o(1) \quad \text{for} \quad k \rightarrow \theta. \end{aligned}$$

If  $T'_{(u)} k$  has a  $F$ -differential with respect to  $u$ , i.e.

$$T'_{(u+k_2)} k_1 - T'_{(u)} k_1 - T''_{(u)} k_1 k_2 = o(\| k_2 \|),$$

the operator  $T''_{(u)}$  is called the second  $F$ -derivative of  $T$ .  $T''_{(u)}$  is a bilinear operator operating on  $k_1$  and  $k_2$ .

d) The operator  $T$  is called completely continuous<sup>3)</sup> or compact if it maps each bounded set of its domain  $D \subset B_1$  in a conditionally compact subset  $S$  of its range  $R \subset B_2$ , that is, in a set  $S \subset R$  each infinite sequence of which contains a subsequence which converges to some element of  $B_2$ .

<sup>1)</sup> For applications it is sometimes more convenient to admit unbounded operators here.

<sup>2)</sup> See, for example, L. V. Kantorovich [2], p. 162.

<sup>3)</sup> See, for example, E. Hille and R. S. Phillips [4], p. 48, or A. E. Taylor [5], p. 274.

For compact operators the Schauder fixed point theorem<sup>1)</sup> holds:

Let the compact operator  $T$  map the convex, closed set  $M \subset D$  into  $M$ :  $TM \subset M$ . Then there exists a fixed point  $u^*$  of  $T$  in  $M$ , that is, a point  $u^* = Tu^*$ .

e) In the following we often consider equations of the form

$$Tu \equiv (T_0 + \Delta T)u = \theta, \quad T, T_0, \Delta T \in (D \rightarrow B_2), \quad (2.1)$$

with an operator  $T$  which lies in a certain neighborhood of  $T_0$  with respect to a sphere  $S(u_0, r)$  of its domain. For the purpose of formulating some neighborhood theorems for those operators we introduce the notation of a  $(u_0, r, a, b)$ -neighborhood, also called an  $\Omega$ -neighborhood, of an operator  $T_0$  with respect to  $S(u_0, r)$ :

*Definition.*  $T$  is said to be lying in an  $\Omega = (u_0, r, a, b)$ -neighborhood of the operator  $T_0$  if and only if

$$\|(T - T_0)u_0\| = \|\Delta Tu_0\| < a, \quad (2.2a)$$

$$\|\Delta Tu - \Delta Tv\| \leq b \|u - v\| \text{ for all } u, v \in S(u_0, r), \quad (2.2b)$$

where  $\Delta T = T - T_0$  and  $S(u_0, r) \subset D_T \cap D_{T_0}$ .

If  $T$  has these properties we briefly write  $T \in \Omega$ .

f) For some proofs we apply the contraction mapping theorem in the following well known form:

*Theorem of contraction mappings*<sup>2)</sup>. Let  $V$  be a contracting operator which maps a closed region  $S \subset B_1$  into itself, i.e.

$$\|Vu - Vv\| \leq l \|u - v\|, \quad l < 1, \quad \text{for } u, v \in S, \quad (2.3)$$

and

$$VS \subset S. \quad (2.4)$$

Then in  $S$ ,  $V$  has exactly one fixed point,  $u = Vu$ .

<sup>1)</sup> J. Schauder [6], for a generalization see A. Tychonoff [7].

<sup>2)</sup> See, for example, J. Weissinger [8] who gave a more general form of this theorem. Without the estimate (2.6), the theorem was used by T. H. Hildebrandt and L. M. Graves [1], p. 133, for the proof of implicit function theorems. Nowadays it is basic for many error estimates in numerical analysis, see L. Collatz [9], p. 36ff. For generalizations see, for example, L. Kantorovich [2], [3], J. Schröder [11], H. Ehrmann [12].

Condition (2.4) is satisfied if (2.3) holds in the sphere

$$S: \|u - u_0\| \leq (1-l)^{-1} \|Vu_0 - u_0\|. \quad (2.5)$$

Moreover,  $u$  is the limit of the sequence  $\{u_n\}$  where

$$u_{n+1} = Vu_n, \quad n = 0, 1, 2, \dots,$$

and there results the estimate

$$\|u - u_{n+1}\| \leq l(1-l)^{-1} \|u_{n+1} - u_n\| \leq l^{n+1}(1-l)^{-1} \|u_1 - u_0\|. \quad (2.6)$$

### 3. THE IMPLICIT FUNCTION THEOREM.

THEOREM 3.1. Let  $T^*$  be an operator with domain  $D \subset B_1$  and range in  $B_2$ , let  $S^* = S(u^*, r^*) \subset D$  and

$$T^* u^* = \theta. \quad (3.1)$$

We assume furthermore that there exists a linear operator  $K$  on  $S^*$  into  $B_2$  with the following properties:

- $\alpha)$   $K$  has a bounded inverse,  $K^{-1}$ , defined on  $B_2$  and
- $\beta)$  There exists a constant  $m < \|K^{-1}\|^{-1}$  such that

$$\|T^*v - T^*u - K(v-u)\| \leq m \|v-u\| \quad \text{for } u, v \in S^*. \quad (3.2)$$

Then there exists an  $\Omega = (u^*, r, a, b)$ -neighborhood of  $T^*$ , such that for all  $T \in \Omega$  the equation

$$Tu = \theta, \quad (3.1a)$$

has a unique solution  $u = u(T)$  in  $S(u^*, r)$ . This solution is continuous in  $T$  at  $T = T^*$  in the sense

$$\|u(T) - u^*\| \rightarrow 0 \quad \text{as} \quad \|Tu^*\| \rightarrow 0. \quad (3.3)$$

In this theorem the operators  $T$  and  $K$  need not be continuous.