ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS *

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CONTENTS

1. Introduction.

This paper presents some existence theorems for the solutions of certain non-linear equations, both local and global theorems. The generality, in particular, of the local theorems is determined largely by the spaces which contain the domain and the range

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of the operator in the equation and the elements the operator depends on.

For example, in the case of the usual implicit function theorem for the solution $u(x)$ of an equation

$$T(x)u \equiv T(x, u) = 0 \tag{1.1}$$

we may successively increase the generality by assuming u, $T(x)u$, and x to be real or complex numbers, vectors, elements of Banach spaces, etc. A very general implicit function theorem for the equation (1.1) in Banach spaces was given by T. H. Hildebrandt and L. M. Graves [1] in 1927.
Here, we are first dealing with equation

$$Tu = 0 \tag{1.2}$$

where u and Tu are supposed to lie in Banach spaces [1]. But we do not assume that the operator T depends on any particular space and we let the solution u depend only on the equation itself: $u = u(T)$. The resulting implicit function theorem is a more general form of the Hildebrandt-Graves theorem and covers theorems known as implicit function, neighborhood, or perturbation theorems, or theorems for the continuous dependence of the solutions upon a parameter. There are many conclusions. It will be shown, for instance, that continuity of the solution $u(T)$ at a " point " $T = T^*$ can be established without assuming continuity of T itself, and under further conditions other local properties of $u(T)$, such as differentiability in the sense of Fréchet, can be proved and the derivatives may be calculated by using only the norms in the Banach spaces of the domain and range of T.

Nevertheless, the main purpose of this paper is to state global existence theorems. Generally, the conditions of the implicit function theorems only suffice for the local existence of a solution in the neighborhood of a given solution of a neighbor equation. But under suitable further conditions the local theorems can be

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1 It would often suffice that the range of T lies in a normed space. But this is not essential here.

applied to global theorems. This is done in different ways and some global theorems are stated.

All theorems in this paper, the local as well as the global theorems, belong to the so-called regular case using certain " linear " methods which insure the existence of a unique solution if the equation or an auxiliar equation in the proof is only disturbed a little. There are no examinations of branch points of solutions, but some global theorems are stated without using complete continuity of the operator which most known theorems do use. In the last three sections complete continuity is needed only for weakening other assumptions and for a few theorems of another kind.

In Section 2 we explain some notations used in this paper, give some definitions of terms which may differ in the literature, and state some well known theorems to be applied in the other sections.

The implicit function theorem mentioned above is given in Section 3. It can be applied immediately to local inverse function theorems. But its assumptions are still not sufficient for the existence of a global inverse of T. Nor is the equation

$$Tu = w, \qquad T \in (B_1 \to B_2), \tag{1.3}$$

solvable for each $w \in B_2$ even if the conditions of the inverse function theorem are satisfied at each point u in B_1. But an additional condition insures the existence. This is the content of Theorems 4.1 and 4.1 *a* of Section 4 which are global inverse function theorems. I presume that the additional condition (the condition γ) in Theorem 4.1) still can be weakened but an easy example, tan $u = w$, u, w real numbers, discussed in Section 5, shows that the assumptions are general enough to cover cases where T is not defined on the whole space B_1 and $Tu = \theta$ does not have a unique solution.

For continuous operators T which satisfy the conditions of Theorem 4.1 we can go into more detail and give an analysis for such operators. This is done in Section 6. The essential result, as it is stated in Theorem 6.1, is that these operators can be split into a number of homeomorphisms of open domains onto B_2. Unlike the linear case, this number can be greater than one, even infinite.

The assumptions of the previous theorems can be partially weakened if we assume that the operator T in (1.2) has a Fréchet-derivative. This is done for the implicit function theorems in Section 7. Under special further assumptions there exist Fréchet-derivatives of certain orders as it is indicated in Section 8. The derivatives of the first and second order are actually calculated. The expressions of the derivatives of higher order of $u\,(T)$ are more complicated in this generality but the considerations of this section show their existence under simple differentiability conditions of T and how to calculate them.

Section 9 gives a global existence theorem using the differentiability of the operator. No complete continuity is required. The essential condition is a boundedness condition on the derivative of a corresponding operator. In particular, in the special case of the application to inverse function theorems only the Fréchet-derived equation has to be investigated. The theorem also states a simple necessary condition that differentiable operators do not assume certain exceptional values. For example, the values $\pm i$ are the only exceptional values of tan x.

Further weakening of the assumptions can be attained if we assume that the equation can be written in the form

$$u = Vu$$

with a completely continuous operator V. As mentioned before, this case is often treated. Nevertheless, the theorems stated in Section 10 and 11 may be useful. In particular, if the operator V is both completely continuous and differentiable, the hypothesis of the theorem are often satisfied. The Theorems 10.1 and 10.2 are local theorems the proofs of which follow immediately from previous theorems. Theorem 10.3 is a global theorem which has a proof similar to the proof of Theorem 4.1 a.

The Theorems 11.1 and 11.2 are of a different kind. They use the Schauder fixed point theorem. The conditions for the operator V are both a boundedness condition in a sphere $\| u - u_0 \| \leqq R$ and a contraction mapping condition in a certain shell $R \leqq \| u - u_0 \| \leqq R_1$, the thickness of which is the smallest possible.

The last section has a quite different character from the previous ones. It contains as an essential result a theorem for the unique solvability of a certain linear equation involving a completely continuous symmetric linear operator. These investigations are of a strictly linear kind, using the theory of eigenvalues of such operators, but applied to special non-linear equations as, for example, non-linear differentiable integral equations of Hammerstein type. They enable us to give explicit conditions on the derivative in order to insure the existence of a solution. This generalizes known existence theorems for such equations.

The literature in the field which is treated here is so extensive that it is impossible to mention all related works.

2. Notations and preliminaries.

a) Throughout this paper the letters B_i, $i = 1, 2, \ldots$, denote Banach spaces with norms $\| u \|_i$, $u \in B_i$, and zero-elements θ_i. For the sake of simplicity we omit the indices on the norms and zero-elements if there is no danger of confusion.

The empty set is denoted by $\emptyset$.

$S(u^*, r)$, means an open, and $\bar{S}(u^*, r)$ *a* closed spherical neighborhood with center u^* and radius r, i.e., the sets

$$\{u: \| u - u^* \| < r\}, \quad \text{respectively} \quad \{u: \| u - u^* \| \leqq r\}.$$

b) We are dealing with (in general non-linear) operators T, V, ... defined on (open) domains D, D_V, ... of Banach spaces and with ranges R, R_V, ... in Banach spaces. We write $T \in (D \to B)$ if $TD = R \subset B$. We assume throughout this paper that the arguments of the operators always lie in the domains of definition if no confusion can occur. The operator I denotes the identity mapping.

For $\quad \dfrac{\| Th \|}{\| Gh \|} \to 0$, or $\dfrac{\| Th \|}{\| Gh \|} \leqq C$ as $h \to \theta$, $h \in D$,