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ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS *

by Hans H. Ehrmann

CONTENTS

1. Introduction.

This paper presents some existence theorems for the solutions of certain non-linear equations, both local and global theorems. The generality, in particular, of the local theorems is determined largely by the spaces which contain the domain and the range

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of the operator in the equation and the elements the operator depends on.

For example, in the case of the usual implicit function theorem for the solution $u(x)$ of an equation

$$T(x)u \equiv T(x, u) = 0 \tag{1.1}$$

we may successively increase the generality by assuming u, $T(x)\,u$, and x to be real or complex numbers, vectors, elements of Banach spaces, etc. A very general implicit function theorem for the equation (1.1) in Banach spaces was given by T. H. Hildebrandt and L. M. Graves [1] in 1927.
Here, we are first dealing with equation

$$Tu = 0 \tag{1.2}$$

where u and Tu are supposed to lie in Banach spaces[1]. But we do not assume that the operator T depends on any particular space and we let the solution u depend only on the equation itself: $u = u(T)$. The resulting implicit function theorem is a more general form of the Hildebrandt-Graves theorem and covers theorems known as implicit function, neighborhood, or perturbation theorems, or theorems for the continuous dependence of the solutions upon a parameter. There are many conclusions. It will be shown, for instance, that continuity of the solution $u(T)$ at a " point " $T = T^*$ can be established without assuming continuity of T itself, and under further conditions other local properties of $u(T)$, such as differentiability in the sense of Fréchet, can be proved and the derivatives may be calculated by using only the norms in the Banach spaces of the domain and range of T.

Nevertheless, the main purpose of this paper is to state global existence theorems. Generally, the conditions of the implicit function theorems only suffice for the local existence of a solution in the neighborhood of a given solution of a neighbor equation. But under suitable further conditions the local theorems can be

[1] It would often suffice that the range of T lies in a normed space. But this is not essential here.

applied to global theorems. This is done in different ways and some global theorems are stated.

All theorems in this paper, the local as well as the global theorems, belong to the so-called regular case using certain " linear " methods which insure the existence of a unique solution if the equation or an auxiliar equation in the proof is only disturbed a little. There are no examinations of branch points of solutions, but some global theorems are stated without using complete continuity of the operator which most known theorems do use. In the last three sections complete continuity is needed only for weakening other assumptions and for a few theorems of another kind.

In Section 2 we explain some notations used in this paper, give some definitions of terms which may differ in the literature, and state some well known theorems to be applied in the other sections.

The implicit function theorem mentioned above is given in Section 3. It can be applied immediately to local inverse function theorems. But its assumptions are still not sufficient for the existence of a global inverse of T. Nor is the equation

$$Tu = w, \qquad T \in (B_1 \to B_2), \tag{1.3}$$

solvable for each $w \in B_2$ even if the conditions of the inverse function theorem are satisfied at each point u in B_1. But an additional condition insures the existence. This is the content of Theorems 4.1 and 4.1 *a* of Section 4 which are global inverse function theorems. I presume that the additional condition (the condition γ) in Theorem 4.1) still can be weakened but an easy example, tan $u = w$, u, w real numbers, discussed in Section 5, shows that the assumptions are general enough to cover cases where T is not defined on the whole space B_1 and $Tu = \theta$ does not have a unique solution.

For continuous operators T which satisfy the conditions of Theorem 4.1 we can go into more detail and give an analysis for such operators. This is done in Section 6. The essential result, as it is stated in Theorem 6.1, is that these operators can be split into a number of homeomorphisms of open domains onto B_2. Unlike the linear case, this number can be greater than one, even infinite.

The assumptions of the previous theorems can be partially weakened if we assume that the operator T in (1.2) has a Fréchet-derivative. This is done for the implicit function theorems in Section 7. Under special further assumptions there exist Fréchet-derivatives of certain orders as it is indicated in Section 8. The derivatives of the first and second order are actually calculated. The expressions of the derivatives of higher order of $u(T)$ are more complicated in this generality but the considerations of this section show their existence under simple differentiability conditions of T and how to calculate them.

Section 9 gives a global existence theorem using the differentiability of the operator. No complete continuity is required. The essential condition is a boundedness condition on the derivative of a corresponding operator. In particular, in the special case of the application to inverse function theorems only the Fréchet-derived equation has to be investigated. The theorem also states a simple necessary condition that differentiable operators do not assume certain exceptional values. For example, the values $\pm i$ are the only exceptional values of tan x.

Further weakening of the assumptions can be attained if we assume that the equation can be written in the form

$$u = Vu$$

with a completely continuous operator V. As mentioned before, this case is often treated. Nevertheless, the theorems stated in Section 10 and 11 may be useful. In particular, if the operator V is both completely continuous and differentiable, the hypothesis of the theorem are often satisfied. The Theorems 10.1 and 10.2 are local theorems the proofs of which follow immediately from previous theorems. Theorem 10.3 is a global theorem which has a proof similar to the proof of Theorem 4.1 *a*.

The Theorems 11.1 and 11.2 are of a different kind. They use the Schauder fixed point theorem. The conditions for the operator V are both a boundedness condition in a sphere $\| u - u_0 \| \leqq R$ and a contraction mapping condition in a certain shell $R \leqq \| u - u_0 \| \leqq R_1$, the thickness of which is the smallest possible.

The last section has a quite different character from the previous ones. It contains as an essential result a theorem for the unique solvability of a certain linear equation involving a completely continuous symmetric linear operator. These investigations are of a strictly linear kind, using the theory of eigenvalues of such operators, but applied to special non-linear equations as, for example, non-linear differentiable integral equations of Hammerstein type. They enable us to give explicit conditions on the derivative in order to insure the existence of a solution. This generalizes known existence theorems for such equations.

The literature in the field which is treated here is so extensive that it is impossible to mention all related works.

2. Notations and preliminaries.

a) Throughout this paper the letters B_i, $i = 1, 2, \ldots$, denote Banach spaces with norms $\| u \|_i$, $u \in B_i$, and zero-elements θ_i. For the sake of simplicity we omit the indices on the norms and zero-elements if there is no danger of confusion.

The empty set is denoted by $\emptyset$.

$S(u^*, r)$, means an open, and $\bar{S}(u^*, r)$ *a* closed spherical neighborhood with center u^* and radius r, i.e., the sets

$$\{u \colon \| u - u^* \| < r\}, \quad \text{respectively} \quad \{u \colon \| u - u^* \| \leqq r\}.$$

b) We are dealing with (in general non-linear) operators T, V, ... defined on (open) domains D, D_V, ... of Banach spaces and with ranges R, R_V, ... in Banach spaces. We write $T \in (D \to B)$ if $TD = R \subset B$. We assume throughout this paper that the arguments of the operators always lie in the domains of definition if no confusion can occur. The operator I denotes the identity mapping.

For $$\frac{\| Th \|}{\| Gh \|} \to 0, \text{ or } \frac{\| Th \|}{\| Gh \|} \leqq C \text{ as } h \to \theta, \ h \in D,$$

we write equivalently

$$Th = o(\| Gh \|), \quad \text{or} \quad Th = O(\| Gh \|).$$

T is continuous at $u \in D$ if $T(u+h) - Tu = o(1)$.

c) By a Fréchet-differential (F-differential) of an operator T at a point $u \in D \subset B$ we understand an expression $T'_{(u)}\, k$ with a (not necessarily bounded [1]) linear operator $T'_{(u)}$ defined on B for which

$$T(u+k) - Tu - T'_{(u)} k = R(u, k) = o(\| k \|).$$

$T'_{(u)}$ is called the Fréchet-derivative (F-derivative).

Let T have a bounded F-derivative for all u of the straight line $u = u_0 + tk$, $0 \leqq t \leqq 1$, then the mean value theorem [2]) holds:

$$\| T(u+k) - Tu \| \leqq \sup_{0 \leqq t \leqq 1} \| T'_{(u+tk)} \| \cdot \| k \|.$$

If T is continuous and differentiable at u, then $T'_{(u)}$ is a continuous operator. This follows from

$$\| T'_{(u)} k \| \leqq \| T(u+k) - Tu - T'_{(u)} k \| + \| T(u+k) - Tu \| \\ = o(1) \quad \text{for} \quad k \to \theta.$$

If $T'_{(u)}\, k$ has a F-differential with respect to u, i.e.

$$T'_{(u+k_2)} k_1 - T'_{(u)} k_1 - T''_{(u)} k_1 k_2 = o(\| k_2 \|),$$

the operator $T''_{(u)}$ is called the second F-derivative of T. $T''_{(u)}$ is a bilinear operator operating on k_1 and k_2.

d) The operator T is called completely continuous [3]) or compact if it maps each bounded set of its domain $D \subset B_1$ in a conditionally compact subset S of its range $R \subset B_2$, that is, in a set $S \subset R$ each infinite sequence of which contains a subsequence which converges to some element of B_2.

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1) For applications it is sometimes more convenient to admit unbounded operators here.

2) See, for example, L. V. Kantorovich [2], p. 162.

3) See ,for example, E. Hille and R. S. Phillips [4], p. 48, or A. E. Taylor [5], p. 274 .

For compact operators the Schauder fixed point theorem [1]) holds:

Let the compact operator T map the convex, closed set $M \subset D$ into M: $TM \subset M$. Then there exists a fixed point u^* of T in M, that is, a point $u^* = Tu^*$.

e) In the following we often consider equations of the form

$$Tu \equiv (T_0 + \Delta T)u = \theta\,, \quad T,\ T_0,\ \Delta T \in (D \to B_2)\,, \tag{2.1}$$

with an operator T which lies in a certain neighborhood of T_0 with respect to a sphere $S(u_0, r)$ of its domain. For the purpose of formulating some neighborhood theorems for those operators we introduce the notation of a (u_0, r, a, b)-neighborhood, also called an Ω-neighborhood, of an operator T_0 with respect to $S(u_0, r)$:

Definition. T is said to be lying in an $\Omega = (u_0, r, a, b)$-neighborhood of the operator T_0 if and only if

$$\| (T - T_0)u_0 \| = \| \Delta T u_0 \| < a\,, \tag{2.2a}$$

$$\| \Delta Tu - \Delta Tv \| \leqq b \| u - v \| \text{ for all } u, v \in S(u_0, r)\,, \tag{2.2b}$$

where $\Delta T = T - T_0$ and $S(u_0, r) \subset D_T \cap D_{T_0}$.

If T has these properties we briefly write $T \in \Omega$.

f) For some proofs we apply the contraction mapping theorem in the following well known form:

Theorem of contraction mappings [2]). Let V be a contracting operator which maps a closed region $S \subset B_1$ into itself, i.e.

$$\| Vu - Vv \| \leqq l \| u - v \|\,, \ l < 1\,, \quad \text{for} \quad u, v \in S\,, \tag{2.3}$$

and

$$VS \subset S\,. \tag{2.4}$$

Then in S, V has exactly one fixed point, $u = Vu$.

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[1]) J. Schauder [6], for a generalization see A. Tychonoff [7].

[2]) See, for example, J. Weissinger [8] who gave a more general form of this theorem. Without the estimate (2.6), the theorem was used by T. H. Hildebrandt and L. M. Graves [1], p. 133, for the proof of implicit function theorems. Nowadays it is basic for many error estimates in numerical analysis, see L. Collatz [9], p. 36ff. For generalizations see, for example, L. Kantorovich [2], [3], J. Schröder [11], H. Ehrmann [12].

Condition (2.4) is satisfied if (2.3) holds in the sphere

$$S\colon \| u - u_0 \| \leqq (1-l)^{-1} \| V u_0 - u_0 \| . \tag{2.5}$$

Moreover, u is the limit of the sequence $\{ u_n \}$ where

$$u_{u+1} = V u_n , \qquad n = 0, 1, 2, \ldots ,$$

and there results the estimate

$$\| u - u_{n+1} \| \leqq l(1-l)^{-1} \| u_{n+1} - u_n \| \leqq l^{n+1} (1-l)^{-1} \| u_1 - u_0 \| . \tag{2.6}$$

3. The implicit function theorem.

Theorem 3.1. Let T^* be an operator with domain $D \subset B_1$ and range in B_2, let $S^* = S(u^*, r^*) \subset D$ and

$$T^* u^* = \theta . \tag{3.1}$$

We assume furthermore that there exists a linear operator K on S^* into B_2 with the following properties:

α) K has a bounded inverse, K^{-1}, defined on B_2 and
β) There exists a constant $m < \| K^{-1} \|^{-1}$ such that

$$\| T^* v - T^* u - K(v-u) \| \leqq m \| v - u \| \quad \text{for} \quad u, v \in S^* . \tag{3.2}$$

Then there exists an $\Omega = (u^*, r, a, b)$-neighborhood of T^*, such that for all $T \in \Omega$ the equation

$$Tu = \theta , \tag{3.1a}$$

has a unique solution $u = u(T)$ in $S(u^*, r)$. This solution is continuous in T at $T = T^*$ in the sense

$$\| u(T) - u^* \| \to 0 \quad \text{as} \quad \| Tu^* \| \to 0 . \tag{3.3}$$

In this theorem the operators T and K need not be continuous.

Proof. Let T lie in a (u^*, r, a, b)-neighborhood of T^* with $r \leqq r^*$. Then by (3.2), with $\Delta T = T - T^*$, we have

$$\begin{aligned} &\| Tv - Tu - K(v-u) \| \leqq \| \Delta Tv - \Delta Tu \| \\ &+ \| T^* v - T^* u - K(v-u) \| \leqq (b+m) \cdot \| v - u \| \qquad (3.4) \\ &\text{for} \quad u, v \in S(u^*, r) \subset S^*, \end{aligned}$$

and the equation

$$u = Vu \equiv K^{-1}(K-T)u, \quad u \in S(u^*, r) = S, \qquad (3.5)$$

is equivalent to (3.1 a), $u \in S$.

For every $b \geqq 0$ with $l = (b+m) \| K^{-1} \| < 1$, (3.4) yields

$$\| Vu - Vv \| = \| K^{-1}[K(u-v) - Tu + Tv] \| \leqq l \| u - v \|, \\ l < 1, \quad \text{for} \quad u, v \in S(u^*, r).$$

If

$$\| Vu^* - u^* \| = \| K^{-1} Tu^* \| < (1-l) r, \qquad (3.6)$$

then the assumptions of the contraction mapping theorem [Section 2 f] are satisfied. Thus, under these conditions, there exists a unique solution $u = u(T)$ in S satisfying the condition

$$\| u - u^* \| \leqq (1-l)^{-1} \| K^{-1} Tu^* \| \leqq (1-l)^{-1} \| K^{-1} \| \cdot \| Tu^* \|. \qquad (3.7)$$

This implies the continuity (3.3).

The inequality (3.6) is satisfied if $T \in \Omega$ with

$$a = [\| K^{-1} \|^{-1} - (b+m)] r.$$

This completes the proof.

This proof also gives quantitative conditions for r, a, b which are sufficient for the existence of a unique and continuous solution u of (3.1 a) in $S(u^*, r)$.

Supplement. The assertion of Theorem 3.1 is true for each Ω-neighborhood of T^* with $0 < r \leqq r^*$ and a, b satisfying

$$a = [\| K^{-1} \|^{-1} - (b+m)] r > 0. \qquad (3.8)$$

Then, for the solution $u = u(T)$ in S, the estimate (3.7) holds.

A unique solution of (3.1 a) in $S(u^*, r)$ also exists for such r and b if (3.6) holds, but in (3.8) the sign " $>$ " cannot be replaced by " $\geqq$ ", nor can the constant a in (3.8) be replaced by any larger one.

The last statement can be proved by simple examples in the one-dimensional case and with an operator T which is linear in $S(u^*, r)$.

4. Inverse function theorems.

Under the conditions of the implicit function Theorem 3.1, the operator T has a local inverse defined in a neighborhood of a point w_0 for which

$$Tu = w. \tag{4.1}$$

has a solution u_0. This inverse has its range in a neighborhood of u_0. For the proof set $T^*u = Tu - w_0$ in Theorem 3.1. However, the conditions of this theorem are still not sufficient for the existence of a solution u of equation (4.1) for all w in B_2 even if T is defined on the whole Banach space B_1 and the conditions are satisfied at each point u of B_1.[1])

However, this actually is not necessary for the existence of at least one solution u of (4.1) for all $w \in B_2$ as is indicated by the following theorem.

Theorem 4.1. Let the operator T, mapping a non-empty domain $D \subset B_1$ into B_2, satisfy the following conditions:

For each $u \in D$ there exist a sphere $S(u, r) \subset D$, a linear operator K, and a constant m such that the following conditions hold:

α) K has a bounded inverse K^{-1} on $TS(u, r)$
β) $\| Tv - T\tilde{v} - K(v - \tilde{v}) \| \leqq m \| v - \tilde{v} \|$ for $v, \tilde{v} \in S(u, r)$
γ) $(\| K^{-1} \|^{-1} - m)\, r \geqq c > 0$ where the constant c is independent of $u \in D$.

1) Example: $Tu \equiv \arctan u = w$, with $B_1 = B_2 = \{\text{real numbers}\}$, is not solvable for all $w \in B_2$, although the conditions of Theorem 3.1 are satisfied at each point $(u, w = \arctan u)$ for $T^*u = Tu - w$.

Then the equation (4.1) has at least one solution for every w in B_2 and each $w_0 \in B_2$ is the center of a sphere $\| w - w_0 \| < c$ for which $u = u(w)$ is continuous and unique in a corresponding neighborhood $S(u_0, r_0)$ with $Tu_0 = w_0$ and $r_0 = r(u_0)$.

Remark. In this theorem it is not required that T be defined for all $u \in B_1$ nor that T be continuous, and it cannot be asserted that T has only one solution for each $w \in B_2$. The example in the footnote (previous page) shows that the condition γ) cannot be improved by deleting $\geqq c$ with constant c independent of u. But γ) can be replaced by other conditions.

THEOREM 4.1 *a.* In Theorem 4.1 the condition γ) can be replaced by

γ') There exists for each $R > 0$ a constant $c = c(R) > 0$ such that

$$(\| K^{-1} \|^{-1} - m) r \geqq c \quad \text{for} \quad \| u \| \leqq R, \text{ and} \tag{4.2}$$

$$\| Tu \| \to \infty \quad \text{as} \quad \| u \| \to \infty \quad \text{and} \quad u \in D. \tag{4.3}$$

Proof of Theorem 4.1. a) Let $u_0 \in D$ and $Tu_0 = w_0$ and let K_0, m_0, r_0 be the corresponding quantities satisfying α), β) and γ) with $S_0 = S(u_0, r_0) \subset D$.
Then by Theorem 3.1 and supplement with $T^*u = Tu - w_0$, $u^* = u_0$, $r = r^* = r_0$ and $b = 0$, it follows that each equation $\tilde{T}u = Tu - w = \theta$ has a unique solution $u(w)$ in S_0 which depends continuously on w provided

$$\| K_0^{-1} \tilde{T} u_0 \| < (1 - m_0 \| K_0^{-1} \|) r_0 = \| K_0^{-1} \| (\| K_0^{-1} \|^{-1} - m_0) r_0 .$$

Because of

$$\| K_0^{-1} \tilde{T} u_0 \| \leqq \| K_0^{-1} \| \cdot \| w_0 - w \| ,$$

and γ) this inequality holds for $\| w - w_0 \| < c$, i.e. (4.1) has a solution $u(w)$ for these w. The solution $u(w)$ is unique and continuous in S_0.

b) Let w_1 be an arbitrary point in B_2. Then the non-empty set Λ of all real λ with $0 \leqq \lambda \leqq 1$ for which the equation

$$Tu - w_0 + \lambda (w_0 - w_1) = \theta ,$$

is solvable is open with respect to the interval [0, 1]. This follows from *a)*. It is also closed, for if $\tilde{\lambda}$ is the supremum of Λ then there exists a point $\lambda^* \in \Lambda$ with $|\tilde{\lambda}^* - \lambda| \, \| w_0 - w_1 \| < c$. Thus it follows from *a)*, if w_0 is replaced by $w_0 - \lambda^* (w_0 - w_1)$, that $\tilde{\lambda} \in \Lambda$. Hence $\Lambda = [0, 1]$ and (4.1) has a solution for all $w \in B_2$.

Proof of Theorem 4.1 a. Let $w_1 \in B_2$ and $u_0 \in D$ with $Tu_0 = w_0$ be given. Then the points $w = w_0 + \lambda (w_1 - w_0)$, $0 \leqq \lambda \leqq 1$, are bounded:

$$\| w \| \leqq \max (\| w_0 \|, \| w_1 \|) = A .$$

Because of γ') there exists a number R with $\| Tu \| > A$ for all u in the set $\{ u \in D : \| u \| \geqq R \}$.[1])

Then the same conclusion as in the proof of Theorem 4.1 with $c = c(R)$ applied to $\| u \| \leqq R$ shows that $Tu = w_1$ is solvable by an element u_1 with $\| u_1 \| < R$ for which the assumptions of Theorem 4.1 with $c = c(R)$ hold. This implies the existence of a sphere $\| w - w_1 \| < c$ with the asserted properties.

5. An example.

The simple example $Tu = \tan u$, given only for illustration purposes, shows that Theorem 4.1 is general enough to cover cases in which either the domain D is not the whole space B_1 or $Tu = w$ does not have a unique solution, although this equation is solvable for all $w \in B_2$.

Let $B_1 = B_2 = B$ be the Banach space of real numbers. Then by Theorem 4.1 the equation

$$Tu \equiv \tan u = w , \quad u, w \in B ,$$

is solvable for all $w \in B$.[2])

Proof. We choose

$$Kv = \frac{v}{\cos^2 u} \quad \text{for} \quad u \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

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1) This set may be empty.
2) This is not true for complex numbers as tan u = i is not solvable.

Then by the mean value theorem and because

$$\frac{d}{du}\frac{1}{\cos^2 u} = \frac{2\sin u}{\cos^3 u},$$

is increasing for increasing

$$u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

it follows that

$$m(u) = \frac{1}{\cos^2(u+r)} - \frac{1}{\cos^2 u} \quad \text{for} \quad 0 \leqq u < \frac{\pi}{2} \quad \text{and} \quad u+r < \frac{\pi}{2}.$$

In the following we restrict ourselves to these u.

From the above we get

$$(\| K^{-1} \|^{-1} - m)r > \left(\frac{1}{\cos^2(u+r)} - \frac{4r}{\cos^3(u+r)}\right) r, \; 0 < r < \frac{\pi}{2} - u.$$

Now choosing r as the smallest positive solution of $r = r(u) = \frac{1}{8}\cos(u+r)$, which implies $u+r < \frac{\pi}{2}$, we get

$$(\| K^{-1} \|^{-1} - m)r > \frac{1}{16\cos(u+r)} > \frac{1}{16}.\,[1]$$

The same is true for $-\frac{\pi}{2} < u < 0$ as can be proved in the same way. Thus the conditions of Theorem 4.1 are valid. In particular γ) is true for $c = \frac{1}{16}$.

6. Inverse function theorems (continued).

As was indicated by the example $\tan u = w$ in the last chapter, the assumptions of the Theorems 4.1 and 4.1 a are not sufficient to insure that the operator T will have an inverse

[1]) Here we use the fact that u is real.

defined on the whole space B_2, i.e. that the equation $Tu = w$ has exactly one solution for each w in B_2. We will now obtain conditions under which the existence of a local inverse implies the existence of a global inverse.

THEOREM 6.1. Let T satisfy the assumptions of Theorem 4.1 and let T be a continuous operator in its domain of definition, D.

Then there exists a finite or infinite number A of open connected domains $D_a \subset D$ with the properties:

$\underset{a \in A}{U} D_a = D$, for each $a \in A$ the restriction T_a of T on D_a is a homeomorphism[1]) of D_a onto B_2, and the sets D_a are mutually disjoint.

Furthermore, if T is defined on the whole Banach space B_1 then T is itself a homeomorphism of B_1 onto B_2.

This theorem implies that under the assumptions there is for each $w \in B_2$ the same finite or infinite number A of solutions of $Tu = w$, and each solution lies in a domain D_a for which the existence of a local inverse implies that of a global one.

Proof. *a)* We first prove the following statement: Let w_1 and w_2 be two points of B_2 with $\| w_1 - w_2 \| < c$ (c from γ) in Theorem 4.1) and let $Tu_1 = w_1$. The existence of at least one such u_1 follows from Theorem 4.1. Furthermore, it is shown that there exists a sphere $S(u_1, r_1) = S_1$ in which the equation $Tu = w$ has a unique solution $u(w)$ for all w with $\| w - w_1 \| < c$. Therefore there exists a unique solution u_2 in S_1 of $Tu = w_2$.

Conversely, let $S(u_2, r_2) = S_2$ the corresponding neighborhood of u_2 in which a unique solution $\tilde{u}$ of $Tu = \tilde{w}$ for $\| \tilde{w} - w_2 \| < c$ exists. Then $w = \tilde{w} \in S(w_1, c) \cap S(w_2, c)$, $u \in S(u_1, r_1)$, $\tilde{u} \in S(u_2, r_2)$, $Tu = w$, $T\tilde{u} = \tilde{w}$ implies $u = \tilde{u}$. If $u \in S_2$ the assertion is true because of the uniqueness of $\tilde{u} = u(\tilde{w})$ in S_2 for $\| \tilde{w} - w_2 \| < c$. Now, let $u \notin S_2$. Then we connect w_2 with w by the straight line $g = w_2 + \lambda (w - w_2)$, $0 \leqq \lambda \leqq 1$, and consider the images C_1 and C_2 of this line in S_1 and S_2, respectively. These images exist and form connected curves $\varphi_i(\lambda) \in S_i$, $i = 1, 2$, using the fact that

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[1]) One-to-one mapping continuous and with continuous inverse.

$g \in S(w_1, c) \cap S(w_2, c)$ in B_2 and applying the theorem that the continuous image of a connected set is connected, which holds in our spaces. We also have $\varphi_i(0) = u_2$, $i = 1, 2$, $\varphi_1(1) = u$, $\varphi_2(1) = \tilde{u}$. In the intersection $S_1 \cap S_2$ the curves C_i coincide because of the uniqueness of $u(w)$, $\tilde{u}(w)$ in S_1, S_2 respectively.

We proceed with increasing λ from u_2 along C_1. Since $u \notin S_2$ there is a first point u^* (with a least $\lambda = \lambda^*$) on C_1 which does not belong to $C_2 \in S_2$. However, in each neighborhood of u^* there are points of C_2. Let $w^* = w_2 + \lambda^*(w - w_2)$, the corresponding point with $Tu^* = w^*$. Then, because of the continuity of C_2, there cannot be another point u on C_2 with $Tu = w^*$, i.e. $u^* \in S_2$ and $C_1 = C_2$ in contradiction to our assumption.

b) Let u_0 be a solution of $Tu = \theta$, which exists by Theorem 4. This theorem also yields a neighborhood $S(u_0, r_0) = S_0$ such that the equation $Tu = w$ has a unique solution $u(w)$ in S_0 for all w with $\|w\| \leqq c - \epsilon$, $0 < \epsilon < c$, and $u(w)$ is continuous there.

We choose a number $R > 0$ arbitrarily large and construct a continuous mapping T_a^{-1} with $T_a^{-1} T = I$ defined for all w with $\|w\| \leqq R$ and with range in a certain domain of B_1. This can be done as follows:

For $\|w\| \leqq c - \epsilon$ the equation $Tu = w$ has a unique and continuous solution, $u(w)$, if u is prescribed to lie in S_0. The (inverse-) images u for these w form a connected closed set in B_1. Let $Tu = w$ be uniquely solvable for all w in the disk $\|w\| \leqq R_1$ by the continuous function $u = u(w)$ and let the set $D_{(R_1)} = \{u = u(w) : \|w\| \leqq R_1\}$ be a connected, closed set containing the point u_0.

Because of the continuity of T the restriction of T to $D_{(R_1)}$ is a one-to-one mapping of $D_{(R_1)}$ onto $\bar{S}(\theta, R_1) \subset B_2$ which is continuous in both directions, i.e. a homeomorphism. In particular, the intersection $S(\tilde{w}, c) \cap \bar{S}(\theta, R_1)$ has its pre-image in the corresponding intersection $S(\tilde{u}, r) \cap D_{(R_1)}$ for each $\tilde{w} \in \bar{S}(\theta, R_1)$ with $T\tilde{u} = \tilde{w}$.

Now we consider the sphere $\| w \| \leqq R_1 + \frac{c}{2} = R_2$. Each w in the shell $R_1 < \| w \| \leqq R_2$ lies in some sphere $\| w - \tilde{w} \| < c$ with $\| \tilde{w} \| \leqq R_1$. We assign to these w the $u = u(w)$ with $Tu = w$ which lies in the corresponding neighborhood $S(\tilde{u}, \tilde{r})$ with $T\tilde{u} = \tilde{w}$. This defines $u(w)$ uniquely. This follows from *a)* since if $\tilde{w}_1$ and $\tilde{w}_2$ are two points in $S(\theta, R_1)$ with $\| w - w_i \| < c$, $i = 1, 2$, then w, w_1 and w_2 lie also in the sphere $S(w^*, c)$ with $w^* = \frac{1}{2}(w_1 + w_2)$ and $\| w^* \| \leqq R_1$. Therefore, it follows from *a)* that our assumptions stated for $\| w \| \leqq R_1$ are true also for $\| w \| \leqq R_1 + \frac{c}{2}$.

Thus, we get a homeomorphism between a certain domain $D_a \subset B_1$ and B_2. Contrary to the case of a linear operator there may be more than one such domain. If there is another solution $u^* \notin D_a$ of $Tu = w^*$ for any $w^* \in B_2$ then by the same construction, with w^* as new center, we obtain another domain D_a^*, and the restriction of T to D_a^* is a homeomorphism on D_a^* onto B_2.

We prove that D_a and D_a^* are disjoint. Let $\tilde{u} \in D_a \cap D_a^*$. Then we connect $\tilde{u}$ with u^* by a curve C^* lying in D_a^*. This curve has an image TC^* in B_2, which is also a curve because of the continuity of T. TC^* has an inverse image $C_a' = T_a^{-1}\, TC^*$ in D_a given by the homeomorphism D_a onto B_2, which is also a curve. C_a' and C^* coincide in $D_a \cap D_a^*$. Let u' be the first point of C^* from $\tilde{u}$ lying on the boundary of D_a. This exists since $u^* \notin D_a$. Then it follows from the continuity of C_a' that $u' \in C_a' \subset D_a$, in contradiction to the openess of D_a. Therefore, D_a and D_a^* are disjoint.

Let T be defined on the whole space B_1. If there is only one domain D_a then the assertion is true. Let there be at least two such domains. Then by a similar consideration connecting two points, $u \in D_a$ and $u^* \in D_a^*$, with the same image by a curve one finds that T cannot be defined on the boundary of such a domain D_a. This contradicts the assumption and completes the proof.

Corollary. If we merely require the assumptions of Theorem 6.1 to be satisfied on a subdomain $D' \subset D$ then all assertions remain true except the last one that T is a homeomorphism of B_1 onto B_2. If there exist two subdomains D_a and $D_a{}^*$ of D' then the assumptions of Theorem 6.1 cannot hold on a whole path P in B_1 connecting D_a and $D_a{}^*$: Either T is not defined everywhere on P as a continuous operator or there does not exist an operator K with bounded inverse satisfying α), β) and γ) of Theorem 4.1.

A similar theorem can be stated using the assumptions of Theorem 4.1 *a* as a basis.

7. Differentiable operators, implicit function theorems.

If the operator T is assumed to be differentiable in the sense of Fréchet (section 2 *c*) then the operator $T'_{(u_0)}$ can be taken as operator K in the previous theorems and similar theorems can be stated.

Theorem 7.1. *a)* Let T_0 be defined on the sphere $S_0 = S(u_0, r_0) \subset B_1$ and let

$$T_0 u_0 = \theta . \tag{7.1}$$

b) Let T_0 have a (not necessarily bounded) derivative $T'_{0(u_0)} = K$ at the point u_0 and let K have a bounded inverse K^{-1} defined on B_2.

c) Assume there are positive numbers $r' \leqq r_0$ and $m = m(r') < \| K^{-1} \|^{-1}$ with

$$\| T_0 (u_0 + u - v) - T_0 u + T_0 v \| \leqq m \| u - v \| , \; u , \; v \in S(u_0 , r') . \tag{7.2}$$

Then an $\Omega = (u_0, r, a, b)$-neighborhood of T_0 exists in which the equation

$$Tu = \theta , \tag{7.3}$$

is uniquely solvable and the solution $u(T)$ is continuous at $T = T_0$. More precisely in Ω we have.

$$\| u(T) - u_0 \| \leqq C \| Tu_0 \| \quad \text{with a constant C.} \tag{7.4}$$

The easy proof follows immediately from Theorem 3.1 and supplement if we observe that, by (7.1),

$$T_0(u_0+k)-Kk = Rk \quad \text{with} \quad Rk = o(\| k \|),$$

and, therefore, because of *b)* and *c)*, there exist positive numbers $r \leqq r'$ and $m_1 < \| K^{-1} \|^{-1}$ with

$$\| K(u-v)-T_0 u+T_0 v \| = \| T_0(u_0+u-v)-T_0 u+T_0 v-R(u-v) \|$$

$$\leqq m_1 \| u-v \| \quad \text{for} \quad u,\ v \in S(u_0, r).$$

Supplement 7.1 a. Conditions *b)* and *c)* can be replaced by the following assumption:

b') At the point u_0, T_0 has a strong derivative[1]) $T'_{0(u_0)} = K$ which has a bounded inverse, i.e. there exists a linear operator K with the property that to every $m > 0$ there is a $r > 0$ such that

$$\| T_0 v-T_0 u-K(v-u) \| \leqq m \| v-u \| \quad \text{if} \quad u,\ v \in S(u_0, r), \tag{7.5}$$

and K has a bounded inverse K^{-1}.

It is easy to show that *b')* implies *b)* and *c)* of Theorem 7.1 or directly α) and β) of Theorem 3.1. Assumption *b')* again holds if we assume T_0 to have a derivative in a whole neighborhood of u_0 and this derivative is continuous and has a bounded inverse. But less is sufficient. More precisely we have the

Supplement 7.1 b. Condition *b')* holds if the following is true:

b'') T_0 has a (not necessarily bounded) derivative $T'_{0(u)}$ in a neighborhood $S(u_0, r)$ of u_0 with the property $T'_{0(u_0)}-T'_{0(u)}$ is bounded and $\| T'_{0(u_0)}-T'_{0(u)} \| \to 0$ as $\| u-u_0 \| \to 0$ and $T'^{-1}_{0(u_0)}$ exists as a bounded operator.

The easy proof follows with $K = T'_{0(u_0)}$ from

$$\| T_0 v-T_0 u-K(v-u) \| \leqq \| T_0 v-T_0 u-T'_{0(u)}(v-u) \| + \| T'_{0(u)}-T'_{0(u_0)} \| \; \| v-u \| .$$

---

1) This notation is introduced by E. B. Leach [13] in connection with an inverse function theorem.

This supplement covers differential operators, for example, which usually are not continuous but have a continuous inverse. For such differential operators which have a derivative satisfying the assumptions *a)* and *b')* or *b'')* the existence of an Ω-neighborhood can only fail at a " point " (T, u) where $T'^{-1}_{(u)}$ does not exist as a bounded linear operator. But the existence of a bounded inverse $T'^{-1}_{(u)}$ for each $u \in B_1$, T being defined everywhere in B_1, is not sufficient to insure that T has an inverse nor that the equation $Tu = w$ is solvable for all $w \in B_2$.

8. On the differentiability of the solution.

In virtue of Theorem 7.1 and supplements the equation $Tu = \theta$ is equivalent to $u = u(T)$ in an Ω-neighborhood of (T_0, u_0) under the above conditions or, in other words, $u(T)$ is a unique function of T defined in Ω by $Tu = \theta$. The conditions yield also the continuity of $u(T)$ in the sense that $u(T)$ tends to u_0 as $\| Tu_0 \| \to 0$ or, more precisely, $\| u(T) - u(T_0) \| \leqq C \| Tu_0 \|$ for some constant C. Therefore,

$$g(u) = o(\| u - u_0 \|) \quad \text{implies} \quad g(u) = o(\| Tu_0 \|), \tag{8.1}$$

for these solutions $u = u(T)$ of $Tu = \theta$.

In order to get the continuity it is sufficient essentially that $\Delta T = T - T_0$ tends to zero at the single point u_0. But for the purpose of calculating a Fréchet-derivative of $u(T)$ we have to know what the behaviour of T is in a neighborhood of u_0 as $\| Tu_0 \| = \| \Delta T u_0 \| \to 0$. According to the definition of the derivative we are looking for a linear operator L such that the expression

$$u(T_0 + \Delta T) - u(T_0) - L \Delta T,$$

tends to zero faster than of order one as $\Delta T \to 0$ in a certain sense. But if we state the formula

$$\begin{aligned} u(T) - u(T_0) &= - T'^{-1}_{0(u_0)} \Delta T u + o(\| u - u_0 \|) \\ &= + T'^{-1}_{0(u_0)} T_0 u + o(\| u - u_0 \|), \end{aligned} \tag{8.2}$$

which follows from

$$T_0 u - T_0 u_0 - T'_{0(u_0)}(u - u_0) = \circ(\| u - u_0 \|),$$

observing that $T_0 u_0 = \theta$ and $Tu = \theta$, we get the difficulty that normally $u(T)$ and $T_0 u$ don't depend linearly on Tu_0 or, equivalently, $\circ(\| u - u_0 \|)$ is not $\circ(\| \Delta Tu \|)$ in general.

Therefore, we make the following natural assumption:

A. We assume that all operators T are differentiable at the point u_0 and that $T'_{(u_0)}$ tends to an operator $\tilde{T}'_{(u_0)}$ for $\| Tu_0 \| \to 0$ such that

$$(\tilde{T}'_{(u_0)} - T'_{(u_0)})(u - u_0) = \circ(\| Tu_0 \|) \quad \text{for} \quad u = u(T),\ u_0 = u(T_0) \tag{8.3}$$

and $\tilde{T}'_{(u_0)}$ has a bounded inverse.

The normal case is $\tilde{T}'_{(u_0)} = T'_{0(u_0)}$, as for example in the usual implicit function theorems. A is more general.

Under this assumption we have the

THEOREM 8.1. Let T_0 satisfy the assumptions of Theorem 7.1 and let Ω be the (u_0, r, a, b)-neighborhood of T_0 in which the equation (7.3) $Tu = \theta$ is uniquely and continuously solvable. Furthermore, we assume that all $T \varepsilon \Omega$ satisfy the differentiability condition *A*.

Then there exists a unique *F*-differential of the solution $u(T)$ of (7.3) at the " point " $T = T_0$ which has the form

$$u'(T_0)\Delta T_0 = -\tilde{T}'^{-1}_{(u_0)} \Delta T_0 u_0, \tag{8.4}$$

where

$$u_0 = u(T_0) \quad \text{and} \quad \Delta T_0 u_0 = (T - T_0) u_0 = Tu_0.$$

Proof. By definition of the *F*-differential of T,

$$\Delta T_0 u_0 = Tu_0 = Tu - T'_{(u_0)}(u - u_0) + \circ(\| u - u_0 \|)$$

$$= -T'_{(u_0)}(u - u_0) + \circ(\| u - u_0 \|),$$

because $Tu = \theta$. Hence it follows by (8.3) and (8.1) that

$$\Delta T_0 u_0 = -\tilde{T}'_{(u_0)}(u - u_0) + \circ(\| Tu_0 \|),$$

or because of the existence of a bounded inverse that

$$u(T) - u(T_0) + \tilde{T}'^{-1}_{(u_0)} \Delta T_0 u_0 = \circ(\| \Delta T_0 u_0 \|), \tag{8.5}$$

which implies (8.4) by definition of the F-differential.

There cannot be more than one such derivative. For let L_1 and L_2 be two linear operators satisfying (8.5). It results from (8.5) with $\lambda \Delta T_0 u_0$ (for fixed $\Delta T_0 u_0$ and real λ) instead of $\Delta T_0 u_0$

$$\| (L_1 - L_2) \Delta T_0 u_0 \| = \varphi(\lambda) \quad \text{with} \quad \varphi(\lambda) \to 0 \quad \text{as} \quad \lambda \to 0,$$

which implies $L_1 = L_2$. This completes the proof.

In the special case $Tu = T^* u - w$, $T_0 u = T^* u - w_0$ and $T^* u_0 = w_0$ the condition A is satisfied with $\tilde{T}'_{(u_0)} = T'_{0(u_0)}$ because of $T'_{(u_0)} = T'_{0(u_0)}$ and assumption *b)* of Theorem 7.1. By writing again T for T^* we get the following inverse function theorem as a corollary:

THEOREM 8.2. *a)* Let T be defined on the sphere $S_0 = S(u_0, r_0) \subset B_1$ and let

$$Tu_0 = w_0 .$$

Furthermore, let the assumptions *b)* and *c)* of Theorem 7.1 be satisfied.

Then T has a local inverse T^{-1} defined in a neighborhood of w_0 and T^{-1} has a bounded derivative at the point w_0:

$$u(w) = T^{-1} w, \qquad u(w_0) = T^{-1} w_0,$$

$$u'(w_0) \Delta w = (T^{-1})'_{(w_0)} \Delta w = (T'_{(u_0)})^{-1} \Delta w, \tag{8.6}$$

with $\Delta w = w - w_0$.

In these theorems it is not required that T and $T'_{(u_0)}$ are continuous although a continuous derivative of the inverse function is asserted. Thus certain differential operators like $F(x, \lambda, u, u', \ldots, u^{(r)})$ plus certain conditions can be treated.

In the special case of an equation

$$Tu \equiv T(x)u \equiv T(x, u) = \theta, \quad T_0 u_0 = T(x_0, u_0) = \theta,$$

with x, u, Tu in Banach spaces we get the usual implicit function theorem with

$$u(T) = u(T(x)) = \varphi(x), \qquad u(T_0) = \varphi(x_0),$$

if we assume that there are F-differentials $T'_{(u)}(x)\,k$, continuous in a neighborhood of (x_0, u_0) and with bounded operator $T'^{-1}_{(u_0)}(x_0)$, and $T'_{(x_0)}(u_0)\,h$. Then

$$\tilde{T}'_{(u_0)} = T'_{(u_0)}(x) \quad \text{and} \quad \varphi'(x_0)\,h = u'(T)\,T'_{(x_0)}(u_0)\,h,$$

and there results the well known formula

$$\varphi'(x_0) = -T'^{-1}_{(u_0)} \cdot T'_{(x_0)}(u_0). \tag{8.7}$$

In order to calculate the second F-differential of the solution $u(T)$ of the equation $Tu = \theta$ at $T = T_0$ we assume that T has a first and a second F-derivative (with respect to u) which are continuous[1] in a neighborhood of u_0. Then also $u'(T)$ is continuous " around T_0 ", i.e. for fixed $h = \Delta^* T$

$$\| u'(T_0 + \Delta T_0)\,h - u'(T_0)\,h \| \to 0 \quad \text{as} \quad \| \Delta T_0 u_0 \| \to 0.$$

Furthermore, according to the case when the operator T depends on the elements of a Banach space B_3, i.e. $Tu = T(x)\,u$, $x \in B_3$, where $\Delta Tu = T(x+h)\,u - T(x)\,u$, we define Δ to be a linear operation:

$$\Delta(T_1 + T_2)\,u = \Delta T_1 u + \Delta T_2 u, \quad \Delta(\lambda Tu) = \lambda \Delta Tu.$$

Then

$$\Delta_1(T + \Delta_2 T)\,u = \Delta_1 Tu + \Delta_1 \Delta_2 Tu,$$

and $\Delta_1 \Delta_2 Tu$ is linear in Δ_1 and Δ_2.

With these natural assumptions the calculation of the second order F-derivative as a bilinear operator is a straight-forward derivation. We use the formula

$$\Delta_1 Tu(T) + T'_{(u)}\,u'(T)\,\Delta_1 T = \theta, \tag{8.8}$$

at the " points " $T = T_0$ and $T = T_0 + \Delta_2 T_0$ and take the

[1]) Less would suffice here, see below.

difference of the two expressions retaining only those terms which are linear in Δ_2. For the sake of brevity we use the following abbreviations:

$$u_0 = u(T_0), \quad T = T_0 + \Delta_2 T_0, \quad u = u(T) = u(T_0 + \Delta_2 T_0),$$
$$\circ_2 = \circ(\| \Delta_2 T_0 u_0 \|).$$

Then we have

$$u(T) = u_0 + u'(T_0) \Delta_2 T_0 + \circ_2,$$
$$k = u'(T) \Delta_1 T = u'(T_0 + \Delta_2 T_0)(\Delta_1 T_0 + \Delta_1 \Delta_2 T_0)$$
$$= u'(T_0 + \Delta_2 T_0) \Delta_1 T_0 + u'(T_0) \Delta_1 \Delta_2 T_0 + \circ_2,$$
$$\Delta_1 T u(T) - \Delta_1 T_0 u_0 = \Delta_1 T_0 u + \Delta_1 \Delta_2 T_0 u - \Delta_1 T_0 u_0$$
$$= \Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_1 \Delta_2 T_0 u_0 + \circ_2, \quad \text{and}$$
$$T'_{0(u)} k = T'_{0(u_0)} k + T''_{0(u_0)} [u'(T_0) \Delta_2 T_0][u'(T_0) \Delta_1 T_0] + \circ_2.$$

Hence

$$T'_{(u)} u'(T) \Delta_1 T = (T_0 + \Delta_2 T_0)'_{(u(T_0 + \Delta_2 T_0))} u'(T_0 + \Delta_2 T_0) \Delta_1 (T_0 +$$
$$\Delta_2 T_0) = [T'_{0(u)} + (\Delta_2 T_0)'_{(u)}] k = T'_{0(u)} k + \Delta_2 T'_{0(u_0)} k$$
$$= T'_{0(u)} k + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0 + \circ_2.$$

Therefore, by (8.8)

$$\theta = \Delta_1 T u + T'_{(u)} u'(T) \Delta_1 T - \Delta_1 T_0 u_0 - T'_{0(u_0)} u'(T_0) \Delta_1 T_0$$
$$= \Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_1 \Delta_2 T_0 u_0 + T'_{0(u_0)} [u'(T) \Delta_1 T_0$$
$$- u'(T_0) \Delta_1 T_0] + T'_{0(u_0)} u'(T_0) \Delta_1 \Delta_2 T_0 + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0$$
$$+ T''_{0(u_0)} [u'(T_0) \Delta_2 T_0][u'(T_0) \Delta_1 T_0] + \circ_2.$$

If we assume as above that $T'_{0(u_0)}$ has a bounded inverse we finally get

$$u'(T_0 + \Delta_2 T_0) \Delta_1 T_0 - u'(T_0) \Delta_1 T_0$$
$$+ T'^{-1}_{0(u_0)} \{\Delta_1 T'_{0(u_0)} u'(T_0) \Delta_2 T_0 + \Delta_2 T'_{0(u_0)} u'(T_0) \Delta_1 T_0$$
$$+ \Delta_1 \Delta_2 T_0 u_0 + T'_{0(u_0)} u'(T_0) \Delta_1 \Delta_2 T_0 + T''_{0(u_0)} [u'(T_0) \Delta_2 T_0]$$
$$[u'(T_0) \Delta_1 T_0]\} + \circ(\| \Delta_2 T_0 u_0 \|).$$

Therefore, the second order differential of the solution $u\,(T)$ of $Tu = \theta$ is given by

$$\begin{aligned} u''(T_0)\,\Delta_2\,T_0\,\Delta_1\,T_0 \;=\; & -T'^{-1}_{0(u_0)}\{\Delta_1\,T'_{0(u_0)}\,u'(T_0)\,\Delta_2\,T_0 \\ & +\Delta_2\,T'_{0(u_0)}\,u'(T_0)\,\Delta_1\,T_0+\Delta_1\,\Delta_2\,T_0\,u_0+T''_{0(u_0)} \\ & [u'(T_0)\,\Delta_2\,T_0][u'(T_0)\,\Delta_1\,T_0]\}-u'(T_0)\,\Delta_1\,\Delta_2\,T_0\,. \end{aligned} \tag{8.9}$$

Here

$$u'(T_0)\,\Delta T_0 \;=\; -T'^{-1}_{(u_0)}\,\Delta T_0\,u_0\,.$$

It is obvious that instead of the boundedness of $T'_{0(u_0)}$ the weaker condition A with $\tilde{T}'_{(u_0)} = T'_{0(u_0)}$ and $Tu_0 = \Delta_2\,T_0\,u_0$ is sufficient for the existence of a differential of second order given by the formula (8.9). The considerations also show the existence of an F-derivative of *n-th* order and how to calculate it if T has F-derivatives up to the order n which are continuous in a neighborhood of u_0 with the possible exception that $T'_{(u_0)}$ satisfies condition A instead of the continuity condition. The uniqueness of the second order derivative can be shown as in the case of the first order derivative.

Example. For the special case

$$Tu \;\equiv\; T(x)\,u \;\equiv\; T(x,u) \;=\; \theta\,, \quad T_0\,u \;\equiv\; T(x_0\,,u)\,, \quad T_0\,u_0 \;=\; \theta\,,$$

we now write $T_x\,(x,\ u)$, $T_u\,(x,\ u)$, $T_{xx}\,(x,\ u)$ etc. for $T'_{(x)}$, $T'_{(u)}$, $T''_{(x)}$ respectively in accordance with the usual notation of partial derivatives of a function of more than one variable.[1])

Assuming x, u, $T\,(x,\ u)$ to be elements of Banach spaces we have with

$$u\,(T) \;=\; u\big(T(x)\big) \;=\; \varphi\,(x)\,,$$

the expressions

$$\varphi'\,(x)\,h \;=\; u'\,(T)\;T_x\,h\,,$$

and

$$\varphi''\,(x)\,h_2\,h_1 \;=\; u''\big(T(x)\big)\,(T_x\,h_2)\,(T_x\,h_1)+u'\big(T(x)\big)\,T_{xx}\,h_2\,h_1\,, \tag{8.10}$$

where the differentials are supposed to be Fréchet-differentials.

Furthermore, we have

$$\Delta_i\,T_0 \;=\; T(x_0+h_i)-T(x_0) \;=\; T_x\,(x_0)\,h_i+\text{o}\,(h_i)\,, \qquad i \;=\; 1,\,2,$$

---

1) The previous notation, however, seems to be more usual in functional analysis

$$
\begin{aligned}
\Delta_2 \Delta_1 T_0 &= \Delta_1 \Delta_2 T_0 \\
&= T(x_0+h_2+h_1) - T(x_0+h_1) - T(x_0+h_2) - T(x_0) \\
&= T_x(x_0+h_1)\, h_2 - T_x(x_0)\, h_2 + \circ(h_2) \\
&= T_{xx}\, h_1\, h_2 + \circ(h_1) + \circ(h_2),
\end{aligned}
$$

and

$$\Delta_i\, T'_{0(u_0)} = T_u(x_0+h_i, u_0) - T_u(x_0, u_0) = T_{xu}\, h_i + \circ(h_i), \quad i = 1, 2.$$

Hence by (8.9) and (8.10), neglecting the terms $\circ(h_i)$, it results

$$
\begin{aligned}
\varphi''(x_0)\, h_2\, h_1 &= -(T_u)^{-1} \{T_{xu}(h_1 [\varphi'(x_0)\, h_2] + h_2 [\varphi'(x_0)\, h_1]) \\
&\quad + T_{xx}\, h_1\, h_2 + T_{uu} [\varphi'(x_0)\, h_1][\varphi'(x_0)\, h_2]\}
\end{aligned}
$$

where the derivatives of T are taken at the point (x_0, u_0) [e.g. $T_u = T_u(x_0, u_0)$] and, for example, $T_{xu}\, hk$ means that the bilinear operator $T_{xu} = T_{xu}(x_0, u_0)$ applies to the elements h and k. Here $\varphi'(x_0)\, h$ can be expressed by $-T_u^{-1}\, T_x\, h$ according to (8.7).

9. A global existence theorem using the differentiability of the operator

In this chapter a method for the proof of the existence of a solution of a non-linear equation

$$Tu = \theta, \tag{9.1}$$

is introduced which may be useful in cases where T has a derivative but cannot be written in the form I-V with completely continuous operator V or in which the complete continuity of V is difficult to show.

THEOREM 9.1. Assume T is a closed[1]) operator defined on an (open) domain $D \subset B_1$ and there has a derivative $T'_{(u)}$ such that $T'_{(u)} - T'_{(v)}$ $(u, v \in D)$ is bounded and continuous[2]) with respect to u. The range of T lies in B_2.

1) See, for example, E. Hille and R. S. Phillips [4], p. 40, or N.I. Achieser and I. M. Glasmann [14], p. 82.

2) We don't require that $T'_{(u)}k$ is continuous with respect to k.

Let T_0 be any operator on $D_0 \supset D$ into B_2 with the properties:

a. $$T_0 u_0 = \theta \quad \text{for some} \quad u_0 \in D. \tag{9.2}$$

b. T_0 has a derivative $T'_{0(u)}$ in D satisfying the same conditions as $T'_{(u)}$

c. The operators
$$T_\lambda = (1-\lambda)\, T_0 + \lambda T, \quad 0 < \lambda < 1,$$
are closed.

Denote
$$U = \{u: T_\lambda u = \theta, \ 0 \leqq \lambda < 1\}.$$

Then either (9.1) has a solution or [1]) the sets

$$S = \{s: s = \frac{\| k \|}{\| T'_{\lambda\,(u)} k \|}, \ k \in B_1, \ u \in U, \ 0 \leqq \lambda < 1\}, \tag{9.3}$$

and

$$V = \{v: v = \| (T - T_0) u \|, \quad u \in U\}, \tag{9.4}$$

are not both bounded.

Proof. Let Λ be the set of all λ in $0 \leqq \lambda \leqq 1$ for which the equation $T_\lambda u = \theta$ has a solution. Then $\Lambda \neq \emptyset$ because $0 \in \Lambda$. Let S be bounded:

$$s \leqq C_1 \quad \text{or} \quad \| T'_{\lambda\,(u)} k \| \geqq \frac{1}{C_1} \| k \|, \quad \frac{1}{C_1} > 0.$$

Therefore [2]), the operator $T'_{\lambda(u)}$ has a bounded inverse $T'^{-1}_{\lambda(u)}$ and

$$\| T'^{-1}_{\lambda(u)} \| \leqq C_1. \tag{9.5}$$

Hence the assumptions of Theorem 7.1, supplement 7.1 *b*, are satisfied. Therefore, it follows that the set Λ is open with respect to [0, 1].

Moreover, Theorem 8.1 says that each " point " $(T_\lambda, u(T_\lambda))$, $u \in U$, has an Ω-neighborhood in which $u = u(T)$ is unique, continuous and differentiable if assumption A of Chapter *8* is satisfied. This is obviously true if we restrict ourselves to

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1) The statements shall not exclude each other, i.e. at least one of them is true.
2) See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

$T_\lambda \in \Omega$. Then the operator $\tilde{T}'_{(u)}$ in (8.3) becomes $T'_{\lambda(u)}$. From this it follows that we can construct a unique and continuously differentiable function $\varphi(\lambda) = u(T_\lambda) \in D$ with $T_\lambda \varphi(\lambda) = \theta$ defined on some interval $0 \leqq \lambda < \tilde{\lambda}$ if we apply the Theorems 7.1 and 8.1 repeatedly. Let $[0, \tilde{\lambda}]$ be the largest interval for which $\varphi(\lambda)$ can be defined by this construction under the assumption that (9.1) is not solvable, i.e. $1 \notin \Lambda$. Obviously $0 < \tilde{\lambda} < 1$ and $\tilde{\lambda} \notin \Lambda$.

Then by (8.7) we have

$$\varphi'(\lambda) = -T'^{-1}_{\lambda(\varphi(\lambda))} T'_{(\lambda)(\varphi(\lambda))} = -T'^{-1}_{\lambda(\varphi(\lambda))}(T - T_0)\, u(T_\lambda) \tag{9.6}$$

for $0 \leqq \lambda < \tilde{\lambda}$. And $\varphi'(\lambda)$ is a bounded linear operator on R^1 into B_1.

Now let $\lambda_\nu < \tilde{\lambda}$, $\nu = 1, 2, \ldots$, be a sequence converging to $\tilde{\lambda}$ and $u_\nu = u(T_{\lambda_\nu}) = \varphi(\lambda_\nu)$ be the solutions of $T_{\lambda_\nu} u = \theta$ as just obtained. Then by the mean value theorem of the differential calculus we have, for $\lambda_\mu > \lambda_\nu$,

$$\| u_\nu - u_\mu \| \leqq \sup_{\lambda_\nu \leqq \lambda \leqq \lambda_\mu} \| \varphi'(\lambda) \| \, | \lambda_\mu - \lambda_\nu | .$$

If we assume that the sets S and V in (9.3), (9.4), respectively, are bounded with bounds C_1 and C_2 then by (9.5) and (9.6)

$$\| u_\nu - u_\mu \| \leqq C_1 C_2 \, | \lambda_\mu - \lambda_\nu | , \qquad \mu, \nu = 1, 2, \ldots .$$

Hence $\{ u_\nu \}$ is a Cauchy sequence and by the completeness of B_1 there exists a limit element $\tilde{u} \in B_1$:

$$\tilde{u} = \lim_{\nu \to \infty} u_\nu .$$

Because $u_\nu \in D$ and $T_{\lambda_\nu} u_\nu = \theta$, $\nu = 1, 2, \ldots$, we have

$$\begin{aligned} \| T_{\tilde{\lambda}} u_\nu \| &= \| (T_{\tilde{\lambda}} - T_{\lambda_\nu}) u_\nu \| = \| (\tilde{\lambda} - \lambda_\nu)(T - T_0) u_\nu \| \\ &\leqq | \tilde{\lambda} - \lambda_\nu | \, \| (T - T_0) u_\nu \| . \end{aligned}$$

By (9.4) and $\lambda_\nu \to \tilde{\lambda}$, $\nu \to \infty$, we have

$$\| T_{\tilde{\lambda}} u_\nu \| \to 0 \quad \text{for} \quad u_\nu \in D, \; u_\nu \to \tilde{u} .$$

Since $T_{\tilde{\lambda}}$ is closed, then

$$\tilde{u} \in D \quad \text{and} \quad T_{\tilde{\lambda}}\, \tilde{u} = \theta .$$

Therefore, Λ also is closed with respect to [0, 1]. Thus $\Lambda = [0.\ 1]$ which completes the proof.

If we choose, in particular, $T_0\, u = Tu - Tu_0$ for some fixed $u_0 \in D$, we get

$$T_\lambda u = Tu - (1-\lambda)\, Tu_0 \quad \text{and} \quad T - T_0 = Tu_0 = \text{const.} \tag{9.7}$$

Thus, all assumptions on T_0 and also the boundedness of the set V are satisfied automatically, and we have the

Corollary 9.1. Assume T is a closed operator defined on an (open) domain $D \subset B_1$ and with range in B_2. Let T have a derivative $T'_{(u)}$ there such that $T'_{(u)} - T'_{(v)}$ is a bounded operator depending continuously on u, (u, $v \in D$).

Then either (9.1) has a solution or the set S in (9.3) is not bounded.

The condition of the boundedness of the set S is equivalent to the condition

$$\inf \{ \| T'_{\lambda\,(u)}\, k \| : \| k \| = 1 ,\ \ k \in B_1 ,\ \ u \in U ,\ \ 0 \leqq \lambda < 1 \} \\ = m > 0 . \tag{9.8}$$

Since $\lambda = 0$ is not excluded there is no statement if $T'_{0(u)}\, k$ is θ for some k; for example, if T_0 is constant. As (9.8) or the boundedness of S is equivalent [1]) also to the existence of a bounded inverse of $T'_{\lambda(u)}$ the existence of a solution of (9.1) can only fail if $T'^{-1}_{\lambda(u)}$ fails to exist as a bounded operator for some $\lambda \in [0, 1]$. The proof of Theorem 9.1 shows that we even can restrict ourselves to examine only $T'^{-1}_{\lambda(u)}$ for $u = \varphi(\lambda)$ or according to formula (8.6) to $(T_\lambda^{-1})'_{(\theta)} = (T'_{\lambda(\varphi(\lambda))})^{-1}$. Thus, writing (9.1) in the form

$$Tu = w_1 , \tag{9.9}$$

and choosing $T_0\, u = Tu - w_0$, $w_0 = Tu_0$, as for (9.7), we get $T_\lambda\, u = Tu - w_0 - \lambda\,(w_1 - w_0)$ and we have the

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1) See, for example, E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

Corollary 9.2. The equation (9.9) with T satisfying the assumptions of Theorem 9.1 has at least one solution if for at least one $u_0 \in D$, with $\varphi(\lambda)$ the same as in the proof of Theorem 9.1, and

$$w(\lambda) = w_0 + \lambda(w_1 - w_0), \tag{9.10}$$

the operators

$$(T'_{(\varphi(\lambda))})^{-1} = (T^{-1})'_{(w(\lambda))}, \quad 0 \leqq \lambda < 1,$$

exist and are bounded uniformly in λ, or equivalently, if $T'^{-1}_{(u_0)}$ exists as a bounded operator and

$$\| (T'_{(\varphi(\lambda))})^{-1} \| = \| (T^{-1})'_{(w(\lambda))} \|,$$

remains finite with increasing λ from 0 to 1.

Example. It is well known that the equation

$$Tz \equiv \tan z = w, \ z, \ w \text{ complex numbers},$$

is not solvable only for $w = \pm i$. Theorem 9.1 immediately shows that the equation is solvable for all $w \neq \pm i$. For

$$(T^{-1})'_{(w)} = \frac{1}{1+w^2},$$

and, with $w_{0_1} = 0 = \tan 0$ and $w_{0_2} = 1 = \tan \frac{\pi}{4}$, all points of the complex number plane can be reached on straight lines (9.10) from either 0 or 1 such that $\dfrac{1}{1+(w(\lambda))^2}$ remains bounded with the only exceptions $w = \pm i$.

10. Completely continuous operators, neighborhood and inverse function theorems.

The assumptions of the theorems can be partially weakened if the non-linear equation can be written in the form

$$u = Vu, \tag{10.1}$$

with a completely continuous operator V. Complete continuity is the most used and most convenient aid for stating existence theorems. Therefore, very may existence theorems use it in their proofs and subtle investigations have been made to show that special operators have this property.[1] Two main ways for using the complete continuity should be emphasized: The fixed point principle based on the Schauder-Tychonoff fixed point theorem[2] and the Leray-Schauder method[3] which is a generalization of the theory of degree of a mapping due to Brouwer. One of the main and nicest results which is important for the applications is the following alternative[4] as basis for a priori estimates:

THEOREM 10.1. If V is defined on a Banach space B with range in B and if V is completely continuous then either (10.1) has a solution or the set $U = \{ u: u = \lambda Vu, \ 0 < \lambda < 1 \}$ is not bounded.

But the boundedness of the set U is, of course, only a sufficient condition and in many cases Theorem 10.1 is not applicable. Moreover, the conditions do not imply the existence of a solution in the neighborhood of a given solution. Therefore, the following theorems, which are analogous to some of the above theorems, may be useful.

As the Fréchet derivative of a completely continuous operator is also completely continuous[5] it is no great restriction of generality if we assume that the linear approximation K of $I-V$, which occurs in Theorems 3.1 and 4.1, has the form $I-L$ with a linear completely continuous operator L. Since a completely continuous operator has only a point spectrum[6], $(I-L)^{-1}$ exists as a bounded linear operator defined on the whole Banach space B_2 if and only if L does not have the eigenvalue[7] 1. Therefore, from Theorem 3.1 there follows immediately the

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1) See, for example, M. A. Krasmosel'skii [15].
2) See section 2f.
3) J. Leray et J. Schauder [16].
4) H. Schaefer [17] grave an elegant proof for this theorem in a more general form.
5) If T is differentiable at u and completely continuous the operators $A(c)$: $A(c)k = \dfrac{T(u+ck)-Tu}{c}$ with real $c > 0$ are also completely continuous and $\| A(c)k - T'_{(u)}k \| = \| k \| \varphi(\| ck \|)$, $\varphi(\delta) \to 0$ as $\delta \to 0$. This implies complete continuity of $T'_{(u)}$. See. for example, A. N. Kolmogorov and S. V. Fomin [18] I, p. 114.
6) See, for example, A. N. Kolmogorov and S. V. Fomin [18] I, p. 117 and 120,
7) λ is an eigenvalue of L if $Lu = \lambda u$ has a non-trivial solution.

Neighborhood Theorem 10.2. Let the equation (10.1) have the solution u_0 and let there exist a completely continuous linear operator L, which does not have the eigenvalue 1, and a number $m < \| (I-L)^{-1} \|^{-1}$ such that

$$\| (V-L)v-(V-L)u \| \leqq m \| v-u \| \quad \text{for} \quad u,v \in S(u_0 , r) , (r > 0).$$

Then an $\Omega = (u_0 , r_0 , a, b)$-neighborhood of $\tilde{T} = I-V$ exists for which the equation

$$Tu = \theta , \quad T \in \Omega , \quad u \in S(u_0 , r_0) , \tag{10.2}$$

is uniquely solvable. The solution $u(T)$ is continuous at $T = \tilde{T}$, i.e.

$$\| u(T)-u_0 \| \to 0 \quad \text{as} \quad \| Tu_0 \| \to 0 .$$

For the special case $Tu = \tilde{T}u - w$ this theorem shows the existence of a local inverse of $\tilde{T}$.

Inverse Function Theorem 10.2 a. If $\tilde{T}u_0 = u_0 - Vu_0 = w_0$ and if the other assumptions of Theorem 10.1 are satisfied then $\tilde{T} = I-V$ has a local inverse, i.e. there exist positive numbers r and b such that

$$u = Vu+w , \quad \| w-w_0 \| < b , \quad \| u-u_0 \| < r ,$$

has a unique solution $u(w)$. Moreover $u(w)$ is continuous at w_0.

These theorems mean, in other words, that the existence of a local neighborhood of $\tilde{T} = I-V$ and u_0 in which the equation (10.1) is uniquely solvable or the existence of a local inverse $\tilde{T}^{-1}$ can only fail if the corresponding linear equation $u = Lu$ is not uniquely solvable.

The above theorems are local theorems insuring the existence of a solution in the neighborhood of a given solution. We now state a global inverse function theorem for the equation

$$u = Vu+w , \tag{10.3}$$

with completely continuous V:

THEOREM 10.3 *a)* Let $T = I - V$ with a completely continuous operator V be defined for all $u \in B_1$.

b) For each $u_0 \in B_1$ let there exist a linear operator $L = L_0$ with bounded operator $(I - L)^{-1}$, defined in a neighborhood of u_0, and a number $m = m_0 < \| (I - L)^{-1} \|^{-1}$ such that

$$\| (V-L) u - (V-L) v \| \leqq m \| u - v \| \quad \text{for} \quad u, v \in S(u_0, r), \, r > 0 .$$

c) Let the sets

$$U(g) = \{u : u = Vu + w, \; w \in g\},$$

for each straight line

$$g = w_0 + \lambda (w_1 - w_0), \; 0 \leqq \lambda \leqq 1, \; w_0, \; w_1 \in B_1,$$

be bounded:

$$\| U(g) \| \leqq C(g) .$$

Then the equation (10.3) has a solution $u = u(w)$ for all $w \in B_1$ and each point $(w, u(w))$ has a (u, r, a, b)-neighborhood.

This theorem is related to Theorem 10.1 concerning the fact that the condition c represents an a priori estimate. However, it is easy to show that the conditions a and c alone are not sufficient for the existence of a solution for each $w \in B_1$.

α) Condition b is satisfied if V has a derivative $V'_{(u)}$ for all $u \in B_1$ and $(I - V'_{(u)})^{-1}$ exists as a bounded operator. This holds true if $V'_{(u)}$ does not have the eigenvalue 1 since $V'_{(u)}$ is completely continuous.

β) Condition c is satisfied if there exists an a priori estimate for the equation (10.3) of the form

$$\| u \| \leqq C \| w \|,$$

or if the condition

$$\| Tu \| = \| (I - V) u \| \to \infty \quad \text{as} \quad \| u \| \to \infty,$$

holds. Therefore, this theorem can be regarded as a certain " generalization " of Theorem 4.1 *a* for completely continuous V. As a matter of fact, the proof is quite analogous.

Proof of Theorem 10.3. α. Let $u_0 \in B_1$ and $Tu_0 = (I - V) u_0 = w_0$. Then from Theorem 3.1 with $K = I - L$ it follows that an open neighborhood of w_0, $\| w - w_0 \| < a$, exists such that (10.3) is solvable for these w.

β. Let $\tilde{w}$ be an arbitrary point of B_1 and let u_0, w_0 be as above. Then the set Λ of all λ, for which

$$Tu = \lambda \tilde{w} + (1 - \lambda) w_0 , \quad 0 \leqq \lambda \leqq 1 ,$$

is solvable, is non-void and open with respect to [0, 1] according to α.

λ. We show that Λ is also closed. Let $\lambda_n \in \Lambda$, $n = 1, 2, \ldots$, be a sequence which converges to λ^*. According to condition c the solutions u_n of $u = Vu + w_n$, $w_n = \lambda_n \tilde{w} + (1 - \lambda_n) w_0$, are bounded. Because of the complete continuity of V there exists a subsequence u_{n_i} such that Vu_{n_i} converges to some element s of the Banach space B_1.

Let $w^* = \lambda^* \tilde{w} + (1 - \lambda^*) w_0$. Then the sequence u_{n_i} converges to $u^* = s + w^*$ in norm. The element u^* is a solution of the equation $u = Vu + w^*$ since

$$\| u_{n_i} - V u_{n_i} - w_{n_i} \| = 0 \quad \text{for} \quad i = 1, 2, 3, \ldots ,$$

and because of the continuity of the norm. Hence $\lambda^* \in \Lambda$ and, therefore, $\Lambda = [0, 1]$.

11. Completely continuous operators, global existence theorems using the Schauder fixed point theorem.

The previous theorems, even the global ones, are derived, roughly speaking, by applying neighborhood theorems and exhausting a domain on the boundary of which the assumptions fail to hold. Here the question suggets itself whether or not corresponding conditions in a shell near the boundary suffice for existence. This indeed is possible for equations

$$u = Vu , \tag{11.1}$$

with a completely continuous operator V. The proof of this statement uses Schauder's fixed point theorem.

THEOREM 11.1. Let V be a completely continuous operator mapping a domain $D \subset B_1$ into B_1 and having the following property.

There exist a point $u_0 \in D$ and non-negative numbers R and C such that

$$\| Vu - u_0 \| \leqq C \quad \text{for} \quad u \in \bar{S}(u_0, R) \subset D. \tag{11.2}$$

If $R < C$ let the additional condition be satisfied:

There is a number $l < 1$ such that

$$\| Vu - Vv \| \leqq l \| u - v \|, \tag{11.3}$$

holds for all u, v in the shell

$$R \leqq \| u - u_0 \| \leqq \frac{C - lR}{1 - l} = R_1 \quad \text{and} \quad \bar{S}_1 = \bar{S}(u_0, R_1) \subset D. \tag{11.4}$$

Then the equation (11.1) has at least one solution in $\| u - u_0 \| \leqq R^*$ where $R^* = R$ in the case $C \leqq R$ and $R^* = R_1$ for $C > R$.

Proof. α) If $C \leqq R$ then $V\bar{S} \subset \bar{S}$ and the fixed point theorem by Schauder [see 2 d] yields the existence of at least one solution $u \in \bar{S}$.

β) Now let $C > R$. Then obviously $R < C \leqq R_1$. Hence $\bar{S} \subset \bar{S}_1$. We prove $V\bar{S}_1 \subset \bar{S}_1$: Let $u \in \bar{S}_1$; then either $u \in \bar{S}$ or $u \in \bar{S}_1 - \bar{S}$. In the first case (11.2) implies $V\bar{S} \subset \bar{S}_1$. In the second case u lies in the shell (11.4). We set

$$v = tu_0 + (1 - t) u \quad \text{with} \quad t = 1 - \frac{R}{\| u - u_0 \|}.$$

It follows that $\| v - u_0 \| = R$. Therefore, by (11.2) and (11.3) we have

$$\| Vu - u_0 \| \leqq \| Vu - Vv \| + \| Vv - u_0 \| \leqq l \| u - v \| + C. \tag{11.5}$$

Furthermore

$$\| u - v \| = \| t(u_0 - u) \| = \| u - u_0 \| - R .$$

Hence by (11.4) and (11.5)

$$\| Vu - u_0 \| \leqq l \| u - u_0 \| - lR + C \leqq \frac{C - lR}{1 - l} = R_1 ,$$

i.e., again $Vu \in \bar{S}_1$. The fixed point theorem completes the proof.

In Theorem 11.1 the estimate $\| u - u_0 \| \leqq R^*$ cannot be improved, and the number R_1 must not be replaced by a smaller one. This can be easily shown.

The application of Theorem 11.1 is easiest if the completely continuous operator V is so constituted that (11.3) with $l < 1$ holds for all u, v outside a certain sphere $S(u_0, R)$.

Theorem 11.1 can be applied to operators of the form

$$Vu = u - (I - K)^{-1} (I - W) u = (I - K)^{-1} (W - K) u .$$

First of all, equation (11.6) shows that along with W and K the operator V is also completely continuous if $(I - K)^{-1}$ exists, i.e., if K has not the eigenvalue 1. From this and Theorem 11.1 there follows easily

THEOREM 11.2. The equation

$$u = Wu ,$$

with a completely continuous operator W has at least one solution if the conditions of Theorem 11.1 for the operator V in (11.6) are satisfied with a completely continuous operator K which has not the eigenvalue 1.

For this the following conditions *a)* or *b)* are sufficient:

a) Let $\| (I - K)^{-1} \| = k$. There exist non-negative numbers c, $m < k^{-1}$, and R such that

$$\| (W - K) u - (I - K) u_0 \| \leqq c \quad \text{for} \quad \| u - u_0 \| \leqq R ,$$

and either $ck \leqq R$ or, if $ck > R$, then

$$\| (W - K) v - (W - K) u \| \leqq m \| v - u \| ,$$

for all u, v in the shell

$$R \leqq \| u - u_0 \| \leqq \frac{k(c - mR)}{1 - km}.$$

b) Let $\| (I - K)^{-1} \| = k$. There exist numbers R and $m < k^{-1}$ such that

$$\| (W - K)v - (W - K)u \| \leqq m \| u - v \| \text{ if } \| u \| > R \text{ and } \| v \| > R.$$

12. Non-linear equations containing a linear completely continuous symmetric operator.

As we have seen in some previous theorems, under certain general conditions, the existence of a solution of an approximating equation or the existence of a solution at all, can fail only if there is no approximating linear operator with bounded inverse or if there is not everywhere such an operator. In the cases when the operators considered are differentiable this means that the derived linear operator does not have a bounded inverse or the derived linear equation fails to have a unique and bounded solution.[1]) It is, therefore, important to have conditions for the existence of a bounded inverse of a corresponding linear operator.

In the case of an operator $I - A$, where A is completely continuous, this is equivalent[2]) to the fact that $u = Au$ has only the solution $u = \theta$, i.e. 1 is not an eigenvalue of A. Here we deal only with such cases and assume our non-linear equation to have the form

$$u = LVu, \tag{12.1}$$

where L is a completely continuous operator and V is an (in general non-linear) operator. This is, indeed, the most usual form of non-linear equations with a completely continuous operator.

Moreover, we now consider the equation (12.1) in a Hilbert space, that is, the operator LV has its domain and range in a

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[1]) This is, of course, typical for the "regular case" of non-linear equations.
[2]) See footnote 2 on page 47.

Hilbert space H. Finally, throughout this section, let L be a symmetric operator.

Under these general assumptions we will give conditions that the derived equation

$$v = LV'_{(u)}v\,, \tag{12.2}$$

have only the trivial solution, $u = \theta$.

To this end we first note some well known statements[1]) on the eigenvalues of a completely continuous symmetric operator: Let A be such an operator defined on a Hilbert space H and with range in H, A being different from the zero-operator.

Then there exists a finite or infinite orthonormal set[2]) of eigenvectors e_i corresponding to real eigenvalues λ_i such that every $u \in H$ can be written uniquely in the form

$$u = \sum_i a_i e_i + u' \quad \text{where} \quad Au' = \theta\,. \tag{12.3}$$

Let us arrange the sequence of eigenvalues as follows:

$$\lambda_{-1} \leqq \lambda_{-2} \leqq \ldots\ldots \leqq \lambda_2 \leqq \lambda_1\,, \tag{12.4}$$

where the λ_n (λ_{-n}), $n \geqq 1$, are positive (negative). One of the two sequences may be empty.

Together with $Au = \lambda u$ we consider the equation

$$u = \kappa A u\,, \quad u \neq \theta\,. \tag{12.5}$$

Then, we have the corresponding sequence[3])

$$\ldots \leqq \kappa_{-2} \leqq \kappa_{-1} < 0 < \kappa_1 \leqq \kappa_2 \leqq \ldots\,, \tag{12.6}$$

of " characteristic values " $k_i = \dfrac{1}{\lambda_i}$ instead of (12.4).

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[1]) See, for example, F. Riesz and B. Sz.-Nagy [19], chapter VI, and A. N. Kolmogorov and S. V. Fomin [18], II, section 27.

[2]) $Ae_i = \lambda_i e_i$, $(e_i, e_k) = \delta_{ik}$.

[3]) The terminology differs in the literature. We define the " eigenvalues " according to the previous sections by $Au = \lambda u$, $u \neq \theta$.

By means of the maximum-minimum principle[1]) we have the independent representations

$$\lambda_1 = \sup_u \{ (Au, u) : \| u \| = 1 \} \qquad \text{and}$$

$$\lambda_n = \inf_{v_i} \sup_u \{ (Au, u) : \| u \| = 1, \ (u, v_i) = 0, \ i = 1, \dots, n-1 \} \tag{12.7}$$

if λ_1 and λ_n, respectively, exist, that is, if the expressions on the right hand side are positive. For λ_{-1} and λ_{-n} we have analogous representations, but the supremum and the infinum must be interchanged.

We now introduce the set P of operators, $p \in P$, which have the following properties:

a) $p \in P$, $u \in H$ implies pu exists and $pu \in H$.

b) All $p \in P$ are linear, continuous, and symmetric,

c) (pu, u) is real for all $u \in H$.

If α is a real number, we write $p < \alpha$, $p \leqq \alpha$, $p > \alpha$, $p \geqq \alpha$ when the corresponding product (pu, u) is $<, \leqq, >, \geqq \alpha (u, u)$, respectively, for all $u \in H$, $u \neq \theta$.

d) If $p \in P$, $p \geqq 0$, then $\sqrt{p} \in P$, $(\sqrt{p})^2 = p$, and $\sqrt{p} < 0 \ (\geqq 0)$ when $p > 0 \ (\geqq 0)$.

Then, obviously, all real numbers α belong to P. It is easy to show that with A and $p \geqq 0$ also the operator $C = \sqrt{p}\, A \sqrt{p}$ is linear, completely continuous, and symmetric. Furthermore, if $p > 0$, then $\sqrt{p} u = \theta$ implies $u = \theta$ and the eigenvalues of Ap and those of $\sqrt{p}\, A \sqrt{p}$ coincide. In fact, $Ap\varphi = \lambda\varphi$ and $\varphi \neq \theta$ imply $\sqrt{p}\, A \sqrt{p}\,\Psi = \lambda\Psi$ with $\Psi = \sqrt{p}\varphi \neq \theta$. The operator $\sqrt{p}\, A \sqrt{p}$ is self-adjoint if A is self-adjoint and $p \geqq 0$, $p \in P$. Therefore, the eigenvalues of Ap are real. On the other hand, if $p > 0$ and $\sqrt{p}\, A \sqrt{p}\,\Psi = \lambda\Psi$ then $\sqrt{p}^{-1}$ exists because $\sqrt{p} u = \theta$ implies $u = \theta$ and with $\varphi = \sqrt{p}^{-1}\,\Psi$ we have $\sqrt{p}\, Ap\varphi = \lambda \sqrt{p}\varphi$ which implies $Ap\varphi = \lambda\varphi$. We have the development

$$u = \sum_i c_i \Psi_i + u' \quad \text{where} \quad \sqrt{p} A \sqrt{p} u' = \theta,$$

1) Courant-Hilbert [20], chapter III, § 3.

and $\{ \Psi_i \}$ is a set of orthonormal eigenvectors of the self-adjoint operator $C = \sqrt{p}\, A \sqrt{p}$.

After these considerations we can prove the following theorem.[1])

THEOREM 12.1. Let A be a linear completely continuous symmetric operator on a Hilbert space H into H, let κ_i be its characteristic values (according to (12.5), (12.6)), and let $p \in P$.

Then the equation

$$u = Apu \,, \tag{12.8}$$

has only the solution $u = \theta$, i.e., $\mu = 1$ is not an eigenvalue of Ap, if one of the following conditions holds:

a) κ_n and κ_{n+1} (κ_{-n} and $\kappa_{-(n+1)}$), $n \geqq 1$ exist and $\kappa_n < p < \kappa_{n+1}$ ($\kappa_{-n} > p > \kappa_{-(n+1)}$) .

b) κ_n (κ_{-n}) exists as the largest positive (smallest negative) characteristic value and $p > \kappa_n$ ($p < \kappa_{-n}$) .

c) There is no positive (negative) characteristic value and $p \geqq 0$ ($p \leqq 0$) .

d) κ_1 (κ_{-1}) exists and $0 \leqq p < \kappa_1$ ($\kappa_{-1} < p \leqq 0$) .

e) $\| p \| < \min\limits_i (| \kappa_i |)$.

Proof. a_1) Let the *n-th* positive characteristic value κ_n of A exist and let $p > \kappa_n > 0$. We show that then the *n-th* positive eigenvalue μ_n of $C = \sqrt{p}\, A \sqrt{p}$ is greater than 1.

Let $\{ e_i \}$ and $\{ \Psi_i \}$ be the sequences of orthogonal and normed eigenvectors of the operators A and C, respectively, corresponding to the eigenvalues $\{ \lambda_i \}$ and $\{ \mu_i \}$, respectively.

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1) In the special case of the boundary value problem $(g(x)y')' + p(x)y = 0$, $y(x_2) = 0$, $y(x_1) = 0$, most of the results follow easily from the Sturm comparison theorem. See, for example, E.A Coddington and N. Levinson [21], chapter 8. In some cases of special equations in which stronger conditions such as $\kappa_n < \alpha_n \leqq p \leqq \alpha_{n+1} < \kappa_{n+1}$ instead of a) hold, the results can be obtained from other well known comparison theorems for eigenvalues, appearing, for instance, in L. Collatz [10], § 9, and F. Riesz and B. Sz.-Nagy [19], section 95.

The system

$$u = \sum_{v=1}^{n} c_v e_v , \quad \| u \| = 1 , \quad (u , \varphi_i) = 0 , \quad i = 1, \ldots, n-1 ,$$

with $\varphi_i = \sqrt{p}^{-1} \Psi_i$, that is

$$\sum_{v=1}^{n} | c_v |^2 = 1 , \quad \sum_{v=1}^{n} c_v (e_v , \varphi_i) = 0 , \quad i = 1, \ldots, n-1 ,$$

is always solvable. For such a u, by (12.4), we have

$$(Au , u) = \sum_{v=1}^{n} \lambda_v | c_v |^2 \geqq \lambda_n .$$

Hence

$$\lambda_n \leqq \sup_u \{ (Au , u) : u = \sum_{v=1}^{n} c_v e_v , \| u \| = 1 , (u , \varphi_i) = 0, \\ i = 1, \ldots , n-1 \} \tag{12.9}$$

$$\leqq \sup_{v \neq \theta} \left\{ \frac{(A\sqrt{p}v , \sqrt{p}v)}{(\sqrt{p}v , \sqrt{p}v)} : (v , \Psi_i) = 0 , \quad i = 1, \ldots , n-1 \right\}$$

since $(\sqrt{p}v, \varphi_i) = (v, \Psi_i)$, $i = 1, \ldots , n-1$, and the first supremum on the right hand side can only become larger if we drop the condition

$$u = \sum_{v=1}^{n} c_v e_v .$$

The assumption $p > \kappa_n = \dfrac{1}{\lambda_n} > 0$ yields

$$\frac{(A\sqrt{p}v , \sqrt{p}v)}{(\sqrt{p}v , \sqrt{p}v)} = \frac{(Cv , v)}{(pv , v)} < \lambda_n \frac{(Cv , v)}{(v , v)} . \tag{12.10}$$

Since the bounded set $\{ c_v \}$ satisfying (12.9) is compact the supremum in (12.9) is actually assumed. Therefore, from (12.9) and (12.10) we get

$$\lambda_n < \lambda_n \sup_v \{ (Cv , v) : \| v \| = 1 , \ (v , \Psi_i) = 0 , \ i = 1, \ldots , n-1 \} \\ = \lambda_n \mu_n \quad \text{or} \quad \mu_n > 1 .$$

a_2) If κ_{n+1} exists and $0 < p < \kappa_{n+1}$ we obtain $\mu_{n+1} < 1$ by a similar argument where the roles of A and C as well as the roles of λ and μ are interchanged.

Thus the equation (12.8) does not have the eigenvalue 1, that is, the theorem holds true for the case *a)* with positive $p \in P$.

b) If κ_{n+1} does not exist but κ_n does, i.e., the right hand side of (12.7) is positive for n but not positive for $n+1$, then, replacing u by $\sqrt{p}u$ with $\kappa_n < p$, we obtain that

$$\inf_{v_i} \sup_{u} \{(\sqrt{p}A\sqrt{p}u, u) : \| u \| = 1, \quad (u, v_i) = 0, \quad i = 1, \dots, n\}$$

also cannot be positive, i.e., $\mu_{n+1} > 0$ does not exist either. From *a_1)* it follows that in this case $\mu_n > 1$ is the smallest positive eigenvalue, i.e. the theorem holds for the case *b)* with positive κ_n and p.

c) If there is no positive eigenvalue then $(Au, u) \leqq 0$ for all u, which obviously implies $(\sqrt{p}\, A \sqrt{p}u, u) = (A \sqrt{p}u, \sqrt{p}u) \leqq 0$ for $p \geqq 0$. Thus 1 is not an eigenvalue.

d) In this case the proof is similar to *a_1)* and *a_2)* if $p \geqq 0$: the largest eigenvalue μ_1 becomes less than one here.

The cases of negative eigenvalues and negative p*'s* can be easily reduced to the positive cases treated above. Let λ_ν^- and k_ν^- be the eigenvalues and characteristic values, respectively, of the operator $-A$. So we have $\lambda_{-n} = -\lambda_n^-$ and the same with κ_ν^-. From $\kappa_{-(n+1)} < p < \kappa_{-n}$ it follows that $\kappa_{n+1}^- > -p > k_n^-$. Because $Ap = -A(-p)$ we can, therefore apply the above results to $-A$ and $-p$ instead of A and p, respectively.

e) We have [1])

$$\min(|\kappa_i|) = \min\left(\frac{1}{|\lambda_i|}\right) = \frac{1}{\max(|\lambda_i|)} = \| A \|^{-1}.$$

Therefore, it follows under the condition *e)* that

$$\| Ap \| \leqq \| A \| \cdot \| p \| < 1 .$$

Hence, 1 is not an eigenvalue.

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[1]) See, for example, N. I. Achieser and I. M. Glasmann [14], p. 47.

This completes the proof.

Theorem 12.1 can be applied to all previous theorems which use the fact that the derived linear equation has only the zero-solution to establish the solvability of the given non-linear equation, provided that this equation can be written in the form

$$u = LVu\,, \tag{12.11}$$

with a linear, completely continuous, and symmetric operator L. In these cases we are able to give explicit conditions on the derivative $V'_{(u)}$ of V as essential conditions for the existence of a solution of (12.11). This derivative plays the part of the operator $p \in P$ in Theorem 12.1. We remember that, in this sense, $V'_{(u)} > \kappa$ is equivalent to $(V'_{(u)}\,\varphi,\ \varphi) > \kappa\,(\varphi,\ \varphi)$ for all $\varphi \in H$, $\varphi \neq \theta$, and the same with $\geqq$, $<$, and $\leqq$. We now give a few examples, first a neighborhood theorem:

Theorem 12.2. Let the product operator LV with a linear completely continuous symmetric operator L and a non-linear continuously differentiable operator V be defined on a Hilbert space H and have its range in H. Let $V'_{(u)}$, $u \in H$, satisfy one of the conditions *a)* through *e)* of Theorem 12.1 with $A = L$ and $V'_{(u)} = p \in P$.

Then for each point $(u_0\,,\ w_0 = u_0 - LVu_0)$ there exists an $\Omega = (u_0\,,\ r,\ a,\ b)$-neighborhood in which the equation

$$u = Tu + w\,, \quad (w + I - T \in \Omega)\,,$$

is uniquely and continuously solvable. In particular, the equation

$$u = LVu + w\,, \tag{12.12}$$

has a unique and continuous solution $u\,(w)$ for w and u in certain spheres about w_0, u_0, respectively, i.e., $I - LV$ has a local inverse there.

The proof follows from Theorem 7.1 and supplements and the fact that a completely continuous operator has only a point spectrum. Therefore, the operator $(I - LV'_{(u)})^{-1}$ is bounded under the assumptions of Theorem 12.2.

The conditions of this theorem are not sufficient for the existence of a solution of (12.12) for each $w \in H$ or, in particular, for $w = \theta$. But as in previous sections, simple additional assumptions assure the existence of a solution of (12.12) for an arbitrary given $w \in H$.

THEOREM 12.3. Let L and V satisfy the conditions of Theorem 12.2 and let one of the following assumptions be fulfilled:

a) For some $u_0 \in H$ and $w_0 = u_0 - LVu_0$ let the set

$$U = \{u : u = LVu + w_0 + \lambda(w - w_0),\ 0 \leqq \lambda < 1\} \qquad (12.13)$$

be bounded.

b) For some $u_0 \in H$ and $w_0 = u_0 - LVu_0$ let the set

$$S = \{s : s = \| k \| \cdot \| (I - LV'_{(u)}) k \|^{-1},\ k \in H,\ u \in U\}, \qquad (12.14)$$

where U is defined in (12.13), be bounded.

Then the equation (12.12) has a solution.

For the proof we set

$$T_\lambda u = (I - LV) u + w_0 + \lambda(w - w_0), \quad 0 \leqq \lambda \leqq 1,$$

and denote by Λ the set of all λ in $[0,\ 1]$ for which $T_\lambda u = \theta$ is solvable. Λ is non-empty because $\lambda = 0$ belongs to Λ. Theorem 12.2 proves Λ is open with respect to $[0,\ 1]$. Λ is also closed. This can be shown in the case *a)* in the same way as in the proof of Theorem 10.3 under λ) where the operator V is to be replaced by LV, and in the case *b)* the proof follows from Theorem 9.1 with $Tu = (I - LV)\, u + w$ and $T_0\, u = (I - LV)\, u + w_0$.

As already remarked in section 9 before corollary 9.2 the boundedness of S, (12.14), is equivalent to the existence of the operators $(I - LV'_{(u)})^{-1}$ as uniformly bounded operators for $u \in U$. The conditions of Theorem 12.1 for $p = V'_{(u)}$ are not strong enough to insure this uniform boundedness with the one exception of condition *c*.

Therefore, we are now going to assume the conditions *a)* through *e)* in the stronger form that p lies in a closed interval for which these conditions hold:

$\bar{a}$*)* κ_n and κ_{n+1} $(\kappa_{-n}$ and $\kappa_{-(n+1)})$, $n \geqq 1$, exist and $\kappa_n < \alpha_n \leqq p \leqq \alpha_{n+1} < \kappa_{n+1}$ $(\kappa_{-n} > \alpha_{-n} \geqq p \geqq \alpha_{-(n+1)} > \kappa_{-(n+1)})$

$\bar{b}$*)* κ_n (κ_{-n}) exists as the largest positive (smallest negative) characteristic value and $p \geqq \alpha_n > \kappa_n$ $(p \leqq \alpha_{-n} < \kappa_{-n})$

$\bar{c}$*)* There is no positive (negative) characteristic value and $p \geqq 0$ $(p \leqq 0)$.

$\bar{d}$*)* κ_1 (κ_{-1}) exists and $0 \leqq p \leqq \alpha_1 < \kappa_1$ $(\kappa_{-1} < \alpha_{-1} \leqq p \leqq 0)$.

$\bar{e}$*)* $\| p \| \leqq \alpha < \min\limits_i (\,|\kappa_i|\,)$.

Here κ_i are the characteristic values of A according to (12.6) and α, α_i are real constants.

Then, instead of Theorem 12.1, we have

THEOREM 12.4. Let A be a linear completely continuous symmetric operator on a Hilbert space H into H, let κ_i be its characteristic values (according to (12.5), (12.6)), and let $p \in P$. Finally, let one of the above conditions $\bar{a}$*)* through $\bar{e}$*)* be satisfied.

Then the inequality

$$|\mu_i - 1| \geqq m > 0\,, \tag{12.14}$$

holds for the eigenvalues μ_i of Ap where m is a constant which does not depend on p but only on the interval $[\alpha_i\,,\ \alpha_j]$ in which p is assumed to lie according to the conditions $\bar{a}$*)* ... $\bar{e}$*)*.

The proof is quite similar to the proof[1]) of Theorem 12.1 and may be left to the reader.

From Theorem 12.4 it follows that, under its assumptions, the norm of $I - Ap$ has a positive lower bound. To prove this fact we assume first that $p > 0$, $p \in P$. Then also $\sqrt{p} > 0$, by definition of P, that is, $\sqrt{p}u = \theta$ implies $u = \theta$, or $\sqrt{p}^{-1}$ exists.[2]) Since $\sqrt{p}^{-1}$ has a bounded inverse [3]), namely $\sqrt{p}$,

$$\| \sqrt{p}^{-1} u \| \geqq k \| u \|\,,\ k > 0\,,\ \text{for all } u \in H\,. \tag{12.15}$$

[1]) For instance, in the first case $\bar{a}$) we get the inequality $\mu_{n+1} \leqq \sigma_{n+1} < 1 < \sigma_n \leqq \mu_n$ where μ_i, σ_{n+1}, σ_n are the eigenvalues of the operators Ap, $A\alpha_{n+1}$, $A\alpha_n$, respectively.

[2]) $\sqrt{p}^{-1}$ is not necessarily in P.

[3]) E. Hille and R. S. Phillips [4], p. 42, Theorem 2.11.6.

Let $\{\Psi_i\}$ be the set of orthonormal eigenvectors of the operator $C = \sqrt{p}A\sqrt{p}$ corresponding to the eigenvalues μ_i of C which are also the eigenvalues of the operator Ap, as already mentioned above. Let u be an arbitrary element in H, $\|u\| = 1$, and $\sqrt{p}u = \sum_i c_i \Psi_i$ where the sum includes the term $c_0 \Psi_0$ in which $C\Psi_0 = \theta$ and $\|\Psi_0\| = 1$.

Then (12.14) and (12.15) yield

$$\|(I - Ap)u\|^2 = \|\sqrt{p}^{-1}(I - C)\sqrt{pu}\|^2 \geqq k^2 \|(I - C)\sqrt{pu}\|^2$$
$$= k^2 \sum_i |c_i|^2 |1 - \mu_i|^2 \geqq k^2 \min(m^2, 1) = \tilde{m}^2 > 0 .$$

Hence

$$\|I - Ap\| \geqq \tilde{m} > 0 .$$

If $p \geqq 0$, i.e. $(pu, u) \geqq 0$ for $u \neq \theta$, then each $u \in H$ is either in the null space, N, of $\sqrt{p}$, i.e. $\sqrt{pu} = \theta$, or it is not. We then consider classes of elements by defining u_1, u_2 to belong to the same class $\bar{u}_c$, briefly $u_1 \equiv u_2$, if and only if $u_1 - u_2 \in N$. Then it follows immediately from $u_1 \equiv u_2$ that $\sqrt{pu_1} \equiv \sqrt{pu_2}$, and vice-versa. Since also $(I - Ap)N = N$ we may regard the operator $\sqrt{p}$ as an operator on the Hilbert space spanned by the congruence classes modulo N, represented by one arbitrary element, $\bar{u}$, of each class. In other words we identify the elements of each class. Thus we have $\sqrt{p\bar{u}} = \theta$ implies $u \in N$, i.e. that $\sqrt{p}^{-1}$ exists, and we can repeat our above argument in the case $\bar{u}_c \neq N$, i.e. $\bar{u} \notin N$.

If $u \in N$ we simply have

$$\|(I - Ap)u\| = \|u\| .$$

The cases $p \leqq 0$ can be treated, as above, by considering the operator $\tilde{A}(-p) = -A(-p)$.

Hence, under the assumptions of Theorem 12.4 we have

$$\|I - Ap\| \geqq c > 0 . \qquad (12.16)$$

These considerations together with Theorem 12.3, setting $L = A$ and $pv = p(u)\, v = V'_{(u)}\, v$, yield the

THEOREM 12.5. Let the product operator LV with a linear completely continuous symmetric operator L and a continuously differentiable operator V be defined on a Hilbert space H and have its range in H.

Let κ_i be the characteristic values of $L = A$ according to (12.5) and (12.6), and let $V'_{(u)}\, v = pu$, $p \in P$, satisfy one of the conditions $\bar{a})$ through $\bar{e})$ (as defined for Theorem 12.4) for each $u \in H$.

Then the equation

$$u = LVu + w,$$

has a solution for each $w \in H$.

This theorem generalizes, for example, some existence theorems for non-linear integral equations of the Hammerstein type, that is, equations of the form [1])

$$u(x) + \int_{\mathfrak{L}} K(x, y) f(y, u(y))\, dy = g(x), \qquad (12.17)$$

where x, y are n-dimensional vectors and $\mathfrak{L}$ is a region in R^n; viz., no definiteness of the kernel K is required and the derivative $f_u(x, u)$ need not be bounded by the least characteristic value k_1.

Example. The problem $-y'' = f(x, y)$, $y(a) = A$, $y(b) = B$, $(b > a)$, is solvable if, for instance, the function f is continuous and continuously differentiable with respect to y in the strip $a \leqq x \leqq b$, $|y| < \infty$, and if $f_y(x, y)$ satisfies there one of the conditions: [2])

$$|f_y(x, y)| \leqq \alpha < \frac{\pi^2}{(b-a)^2}; \text{ or } f_y < 0;$$

or

$$\frac{n^2 \pi^2}{(b-a)^2} < \alpha_n \leqq f_y(x, y) \leqq \alpha_{n+1} < \frac{(n+1)^2 \pi^2}{(b-a)^2}.$$

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1) A. Hammerstein [22], see also F. G. Tricomi [23], section 4.6.

2) The known theorems usually cover only the first two cases of this special example. See F. Lettenmeyer [24] and H. Epheser [25]. These papers are more general in another direction.

The proof follows immediately from Theorem 12.5 by writing the problem in the form (12.17). In this case the operator L happens to be definite. But this is not required or used in the proof.

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