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9. THE RELATED EQUATION.

We are prepared now to make the construction toward which this entire discussion has been directed.

Consider the equation

$$L^*(u) = 0. \tag{9.1}$$

with

$$L^*(u) = \frac{1}{T} \begin{bmatrix} m^*(\eta_1) & - & - & - & - & m^*(\eta_p) & m^*(u) \\ Dm^*(\eta_1) & - & - & - & - & Dm^*(\eta_p) & Dm^*(u) \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ D^{p-1}m^*(\eta_1) & - & - & - & - & D^{p-1}m^*(\eta_p) & D^{p-1}m^*(u) \\ l^*(m^*(\eta_1)) & - & - & - & - & l^*(m^*(\eta_p)) & l^*(m^*(u)) \end{bmatrix}. \tag{9.2}$$

$T$  being the determinant given in (8.4). This is clearly a differential equation of the  $n^{th}$  order in  $u$ , for which each one of the functions  $y_j(z, \lambda)$  and  $\eta_i(z, \lambda)$  is a solution. For if  $\eta_i$  is substituted for  $u$  two of the columns of the determinant (9.2) are the same, and if  $u$  is replaced  $y_j$  every element of the last column vanishes. Because the  $n$  solutions thus produced are linearly independent the solutions of the equation (9.1) are completely known.

The co-factor of the element  $l^*(m(u))$  in the formula (9.2) is the determinant  $T$ . The expansion of the formula thus gives it the aspect

$$L^*(u) = l^*(m^*(u)) - \sum_{v=1}^p \frac{T_v}{T} D^{p-v} m^*(u), \tag{9.3}$$

where  $T_v$  is the determinant that is obtainable from the formula (8.4) by replacing its elements  $D^{p-v} m^*(\eta_j)$  by  $l^*(m^*(\eta_j))$ .

From the formula (8.5) it is seen that

$$l^*(m^*(\eta_j)) = \lambda^n \sum_{v=1}^p \frac{\tau_v(z, \lambda)}{\lambda^r} \cdot \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}} \tag{9.4}$$

with

$$\tau_v(z, \lambda) = \sum_{k=0}^p \bar{\beta}_k(z, \lambda) \sigma_{v,r}^{(p-k)}(z, \lambda). \tag{9.5}$$

The replacements which change  $T$  to  $T_\nu$  are thus seen to be ones which replace

$$\lambda^{n-\nu} \left\{ \delta_{p-\nu, j} + \frac{\sigma_{j, r}^{(p-\nu)}}{\lambda^r} \right\} \text{ by } \lambda^n \frac{\tau_\nu}{\lambda^r}.$$

It follows that

$$\frac{T_\nu}{T} = \lambda^\nu \frac{\theta_\nu(z, \lambda)}{\lambda^r},$$

with some function  $\theta_\nu(z, \lambda)$  which is bounded over the  $z$  and  $\lambda$  domains. This gives to the relation (9.3) the form

$$L^*(u) = l^*(m^*(u)) - \frac{1}{\lambda^r} \sum_{\nu=1}^p \lambda^\nu \theta_\nu D^{p-\nu} m^*(u). \quad (9.7)$$

With the substitution of the expression for  $D^{p-\nu} m^*(u)$ , as it may be obtained from (4.3) by writing  $\bar{\gamma}_{i-s}$  in the place of  $\gamma_{i-s}$ , it is found that

$$L^*(u) = l^*(m^*(u)) - \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \omega_j(z, \lambda) D^{n-j} u, \quad (9.8)$$

with

$$\omega_j(z, \lambda) = \sum_{\nu=1}^p \sum_{s=0}^p \lambda^{-s} \binom{p-\nu}{s} \theta_\nu D^s \bar{\gamma}_{\mu-\nu-s}.$$

A comparison of this with the earlier result (6.6) shows that

$$L^*(u) = L(u) - \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \{ \varepsilon_j(z, \lambda) + \omega_j(z, \lambda) \} D^{n-j} u. \quad (9.9)$$

The equation (9.1), whose solutions are completely known, thus has coefficients which differ from those of the given equation (2.1) only by terms that are of at least the  $r^{\text{th}}$  degree in  $1/\lambda$ . It is, therefore, by definition, a related equation.

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