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one, and accordingly admits of an analytic solution for  $(a^{(0)})_0$  provided the matrix multiplier of this vector on the left is non-singular. This condition is assured by the relation (3. 4).

Now we may proceed by induction. Assuming that the vectors  $(a^{(0)})_j$  for  $j = 1, 2, \dots, (v-1)$ , have been determined and are analytic, the right-hand member of the equation (7. 7) is known. As in the case  $v = 0$ , so now, the equation is analytically solvable. The solutions for the successive values  $v = 0, 1, 2, \dots, (r-1)$ , yield the coefficients (6. 7) for which the functions  $\eta_i(z, \lambda)$ , as given by the formulas (6. 8), fulfill the relations (6. 5).

8. ON LINEAR INDEPENDENCE.

With the functions  $a_j^{(0)}(z, \lambda)$  now at hand, we have at our disposal the  $n$  known functions  $y_j(z, \lambda)$ ,  $j = 1, 2, \dots, q$ , which are the solutions of the differential equation (6. 3), and  $\eta_i(z, \lambda)$ ,  $i = 1, 2, \dots, p$ , which are given by the formulas (6. 8). We shall show that these functions are linearly independent.

Let the Wronskians of the entire set and of the respective sub-sets be denoted respectively by  $W_n, W_q(y)$  and  $W_p(\eta)$ . If the usual form

$$W_n = \begin{bmatrix} y_1 & - & - & - & y_q & \eta_1 & - & - & - & \eta_p \\ Dy_1 & - & - & - & Dy_q & D\eta_1 & - & - & - & D\eta_p \\ - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ D^{n-1}y_1 & - & - & - & D^{n-1}y_q & D^{n-1}\eta_1 & - & - & - & D^{n-1}\eta_p \end{bmatrix} \quad (8. 1)$$

is modified by adding to each of the last  $p$  rows suitable multiples of the preceding ones, the formula can be made to appear thus

$$= \begin{bmatrix} y_1 & - & - & - & y_q & \eta_1 & - & - & - & \eta_p \\ Dy_1 & - & - & - & Dy_q & D\eta_1 & - & - & - & D\eta_p \\ - & - & - & - & - & - & - & - & - & - \\ D^{q-1}y_1 & - & - & - & D^{q-1}y_q & D^{q-1}\eta_1 & - & - & - & D^{q-1}\eta_p \\ m^*(y_1) & - & - & - & m^*(y_q) & m^*(\eta_1) & - & - & - & m^*(\eta_p) \\ Dm^*(y_1) & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ D^{p-1}m^*(y_1) & - & - & - & D^{p-1}m^*(y_q) & D^{p-1}m^*(\eta_1) & - & - & - & D^{p-1}m^*(\eta_p) \end{bmatrix} \quad (8. 2)$$

In this, however, each of the elements occupying a position in one of the first  $q$  columns and in one of the last  $p$  rows is zero. The formula therefore reduces at once to

$$W_n = W_q(y) T, \quad (8.3)$$

with

$$T = \begin{bmatrix} m^*(\eta_1) & - & - & - & - & m^*(\eta_p) \\ Dm^*(\eta_1) & - & - & - & - & Dm^*(\eta_p) \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ D^{p-1}m^*(\eta_1) & - & - & - & - & D^{p-1}m^*(\eta_p) \end{bmatrix}. \quad (8.4)$$

Now  $m^*(\eta_j)$  is given by the formula (6.15). If this is repeatedly differentiated, and at each step the element  $D^p v_j$  is eliminated by use of the equation (6.1), the results are the formulas

$$D^i m^*(\eta_j) = \lambda^q D^i v_j + \lambda^{q+i-r} \sum_{\mu=0}^p \lambda^{1-\mu} \sigma_{\mu,r}^{(i)} D^{\mu-1} v_j, \quad i = 0, 1, 2, \dots \quad (8.5)$$

We may write this also, with the use of the symbol  $\delta_{i,j}$  to denote 1 when  $j = i$  and 0 when  $j \neq i$ , in the form

$$D^{i-1} m^*(\eta_j) = \lambda^{q+i-1} \sum_{\mu=1}^p \left\{ \delta_{i,\mu} + \frac{\sigma_{\mu,r}^{(i-1)}}{\lambda^r} \right\} \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}}. \quad (8.6)$$

This shows, now, at once, that the determinant  $T$  can be factored, thus

$$T = \lambda^{pq} E W_p(v) \quad (8.7)$$

in which  $E$  is the determinant whose element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column is indicated thus

$$E = \left| \delta_{i,j} + \frac{\sigma_{j,r}^{(i-1)}}{\lambda^r} \right|. \quad (8.8)$$

It is clear that  $E$  differs from 1 by terms of at least the degree  $r$  in  $1/\lambda$ . Since  $W_p(v)$  and  $W_q(y)$  are non-vanishing, it follows from (8.3) and (8.7) that the same is true of  $W_n$ .

9. THE RELATED EQUATION.

We are prepared now to make the construction toward which this entire discussion has been directed.

Consider the equation

$$L^*(u) = 0. \tag{9.1}$$

with

$$L^*(u) = \frac{1}{T} \begin{bmatrix} m^*(\eta_1) & - & - & - & - & - & m^*(\eta_p) & m^*(u) \\ Dm^*(\eta_1) & - & - & - & - & - & Dm^*(\eta_p) & Dm^*(u) \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ D^{p-1}m^*(\eta_1) & - & - & - & - & - & D^{p-1}m^*(\eta_p) & D^{p-1}m^*(u) \\ l^*(m^*(\eta_1)) & - & - & - & - & - & l^*(m^*(\eta_p)) & l^*(m^*(u)) \end{bmatrix}. \tag{9.2}$$

$T$  being the determinant given in (8.4). This is clearly a differential equation of the  $n^{th}$  order in  $u$ , for which each one of the functions  $y_j(z, \lambda)$  and  $\eta_i(z, \lambda)$  is a solution. For if  $\eta_i$  is substituted for  $u$  two of the columns of the determinant (9.2) are the same, and if  $u$  is replaced  $y_j$  every element of the last column vanishes. Because the  $n$  solutions thus produced are linearly independent the solutions of the equation (9.1) are completely known.

The co-factor of the element  $l^*(m(u))$  in the formula (9.2) is the determinant  $T$ . The expansion of the formula thus gives it the aspect

$$L^*(u) = l^*(m^*(u)) - \sum_{v=1}^p \frac{T_v}{T} D^{p-v} m^*(u), \tag{9.3}$$

where  $T_v$  is the determinant that is obtainable from the formula (8.4) by replacing its elements  $D^{p-v} m^*(\eta_j)$  by  $l^*(m^*(\eta_j))$ .

From the formula (8.5) it is seen that

$$l^*(m^*(\eta_j)) = \lambda^n \sum_{v=1}^p \frac{\tau_v(z, \lambda)}{\lambda^r} \cdot \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}} \tag{9.4}$$

with

$$\tau_v(z, \lambda) = \sum_{k=0}^p \bar{\beta}_k(z, \lambda) \sigma_{v,r}^{(p-k)}(z, \lambda). \tag{9.5}$$

The replacements which change  $T$  to  $T_v$  are thus seen to be ones which replace

$$\lambda^{n-v} \left\{ \delta_{p-v, j} + \frac{\sigma_{j, r}^{(p-v)}}{\lambda^r} \right\} \text{ by } \lambda^n \frac{\tau_v}{\lambda^r}.$$

It follows that

$$\frac{T_v}{T} = \lambda^v \frac{\theta_v(z, \lambda)}{\lambda^r},$$

with some function  $\theta_v(z, \lambda)$  which is bounded over the  $z$  and  $\lambda$  domains. This gives to the relation (9.3) the form

$$L^*(u) = l^*(m^*(u)) - \frac{1}{\lambda^r} \sum_{v=1}^p \lambda^v \theta_v D^{p-v} m^*(u). \quad (9.7)$$

With the substitution of the expression for  $D^{p-v} m^*(u)$ , as it may be obtained from (4.3) by writing  $\bar{\gamma}_{i-s}$  in the place of  $\gamma_{i-s}$ , it is found that

$$L^*(u) = l^*(m^*(u)) - \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \omega_j(z, \lambda) D^{n-j} u, \quad (9.8)$$

with

$$\omega_j(z, \lambda) = \sum_{v=1}^p \sum_{s=0}^p \lambda^{-s} \binom{p-v}{s} \theta_v D^s \bar{\gamma}_{\mu-v-s}.$$

A comparison of this with the earlier result (6.6) shows that

$$L^*(u) = L(u) - \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \{ \varepsilon_j(z, \lambda) + \omega_j(z, \lambda) \} D^{n-j} u. \quad (9.9)$$

The equation (9.1), whose solutions are completely known, thus has coefficients which differ from those of the given equation (2.1) only by terms that are of at least the  $r^{\text{th}}$  degree in  $1/\lambda$ . It is, therefore, by definition, a related equation.

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