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one, and accordingly admits of an analytic solution for $(a^{(0)})_0$ provided the matrix multiplier of this vector on the left is non-singular. This condition is assured by the relation (3. 4).

Now we may proceed by induction. Assuming that the vectors $(a^{(0)})_j$ for $j = 1, 2, \dots, (v-1)$, have been determined and are analytic, the right-hand member of the equation (7. 7) is known. As in the case $v = 0$, so now, the equation is analytically solvable. The solutions for the successive values $v = 0, 1, 2, \dots, (r-1)$, yield the coefficients (6. 7) for which the functions $\eta_i(z, \lambda)$, as given by the formulas (6. 8), fulfill the relations (6. 5).

8. ON LINEAR INDEPENDENCE.

With the functions $a_j^{(0)}(z, \lambda)$ now at hand, we have at our disposal the n known functions $y_j(z, \lambda)$, $j = 1, 2, \dots, q$, which are the solutions of the differential equation (6. 3), and $\eta_i(z, \lambda)$, $i = 1, 2, \dots, p$, which are given by the formulas (6. 8). We shall show that these functions are linearly independent.

Let the Wronskians of the entire set and of the respective sub-sets be denoted respectively by $W_n, W_q(y)$ and $W_p(\eta)$. If the usual form

$$W_n = \begin{bmatrix} y_1 & - & - & - & y_q & \eta_1 & - & - & - & \eta_p \\ Dy_1 & - & - & - & Dy_q & D\eta_1 & - & - & - & D\eta_p \\ - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ D^{n-1}y_1 & - & - & - & D^{n-1}y_q & D^{n-1}\eta_1 & - & - & - & D^{n-1}\eta_p \end{bmatrix} \quad (8. 1)$$

is modified by adding to each of the last p rows suitable multiples of the preceding ones, the formula can be made to appear thus

$$= \begin{bmatrix} y_1 & - & - & - & y_q & \eta_1 & - & - & - & \eta_p \\ Dy_1 & - & - & - & Dy_q & D\eta_1 & - & - & - & D\eta_p \\ - & - & - & - & - & - & - & - & - & - \\ D^{q-1}y_1 & - & - & - & D^{q-1}y_q & D^{q-1}\eta_1 & - & - & - & D^{q-1}\eta_p \\ m^*(y_1) & - & - & - & m^*(y_q) & m^*(\eta_1) & - & - & - & m^*(\eta_p) \\ Dm^*(y_1) & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ D^{p-1}m^*(y_1) & - & - & - & D^{p-1}m^*(y_q) & D^{p-1}m^*(\eta_1) & - & - & - & D^{p-1}m^*(\eta_p) \end{bmatrix} \quad (8. 2)$$

In this, however, each of the elements occupying a position in one of the first q columns and in one of the last p rows is zero. The formula therefore reduces at once to

$$W_n = W_q(y) T, \quad (8.3)$$

with

$$T = \begin{bmatrix} m^*(\eta_1) & - & - & - & - & m^*(\eta_p) \\ Dm^*(\eta_1) & - & - & - & - & Dm^*(\eta_p) \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ D^{p-1}m^*(\eta_1) & - & - & - & - & D^{p-1}m^*(\eta_p) \end{bmatrix}. \quad (8.4)$$

Now $m^*(\eta_j)$ is given by the formula (6.15). If this is repeatedly differentiated, and at each step the element $D^p v_j$ is eliminated by use of the equation (6.1), the results are the formulas

$$D^i m^*(\eta_j) = \lambda^q D^i v_j + \lambda^{q+i-r} \sum_{\mu=0}^p \lambda^{1-\mu} \sigma_{\mu,r}^{(i)} D^{\mu-1} v_j, \quad i = 0, 1, 2, \dots \quad (8.5)$$

We may write this also, with the use of the symbol $\delta_{i,j}$ to denote 1 when $j = i$ and 0 when $j \neq i$, in the form

$$D^{i-1} m^*(\eta_j) = \lambda^{q+i-1} \sum_{\mu=1}^p \left\{ \delta_{i,\mu} + \frac{\sigma_{\mu,r}^{(i-1)}}{\lambda^r} \right\} \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}}. \quad (8.6)$$

This shows, now, at once, that the determinant T can be factored, thus

$$T = \lambda^{pq} E W_p(v) \quad (8.7)$$

in which E is the determinant whose element in the i^{th} row and j^{th} column is indicated thus

$$E = \left| \delta_{i,j} + \frac{\sigma_{j,r}^{(i-1)}}{\lambda^r} \right|. \quad (8.8)$$

It is clear that E differs from 1 by terms of at least the degree r in $1/\lambda$. Since $W_p(v)$ and $W_q(y)$ are non-vanishing, it follows from (8.3) and (8.7) that the same is true of W_n .

9. THE RELATED EQUATION.

We are prepared now to make the construction toward which this entire discussion has been directed.

Consider the equation

$$L^*(u) = 0. \tag{9.1}$$

with

$$L^*(u) = \frac{1}{T} \begin{bmatrix} m^*(\eta_1) & - & - & - & - & m^*(\eta_p) & m^*(u) \\ Dm^*(\eta_1) & - & - & - & - & Dm^*(\eta_p) & Dm^*(u) \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ D^{p-1}m^*(\eta_1) & - & - & - & - & D^{p-1}m^*(\eta_p) & D^{p-1}m^*(u) \\ l^*(m^*(\eta_1)) & - & - & - & - & l^*(m^*(\eta_p)) & l^*(m^*(u)) \end{bmatrix}. \tag{9.2}$$

T being the determinant given in (8.4). This is clearly a differential equation of the n^{th} order in u , for which each one of the functions $y_j(z, \lambda)$ and $\eta_i(z, \lambda)$ is a solution. For if η_i is substituted for u two of the columns of the determinant (9.2) are the same, and if u is replaced y_j every element of the last column vanishes. Because the n solutions thus produced are linearly independent the solutions of the equation (9.1) are completely known.

The co-factor of the element $l^*(m(u))$ in the formula (9.2) is the determinant T . The expansion of the formula thus gives it the aspect

$$L^*(u) = l^*(m^*(u)) - \sum_{v=1}^p \frac{T_v}{T} D^{p-v} m^*(u), \tag{9.3}$$

where T_v is the determinant that is obtainable from the formula (8.4) by replacing its elements $D^{p-v} m^*(\eta_j)$ by $l^*(m^*(\eta_j))$.

From the formula (8.5) it is seen that

$$l^*(m^*(\eta_j)) = \lambda^n \sum_{v=1}^p \frac{\tau_v(z, \lambda)}{\lambda^r} \cdot \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}} \tag{9.4}$$

with

$$\tau_v(z, \lambda) = \sum_{k=0}^p \bar{\beta}_k(z, \lambda) \sigma_{v,r}^{(p-k)}(z, \lambda). \tag{9.5}$$

The replacements which change T to T_v are thus seen to be ones which replace

$$\lambda^{n-v} \left\{ \delta_{p-v, j} + \frac{\sigma_{j, r}^{(p-v)}}{\lambda^r} \right\} \text{ by } \lambda^n \frac{\tau_v}{\lambda^r}.$$

It follows that

$$\frac{T_v}{T} = \lambda^v \frac{\theta_v(z, \lambda)}{\lambda^r},$$

with some function $\theta_v(z, \lambda)$ which is bounded over the z and λ domains. This gives to the relation (9.3) the form

$$L^*(u) = l^*(m^*(u)) - \frac{1}{\lambda^r} \sum_{v=1}^p \lambda^v \theta_v D^{p-v} m^*(u). \quad (9.7)$$

With the substitution of the expression for $D^{p-v} m^*(u)$, as it may be obtained from (4.3) by writing $\bar{\gamma}_{i-s}$ in the place of γ_{i-s} , it is found that

$$L^*(u) = l^*(m^*(u)) - \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \omega_j(z, \lambda) D^{n-j} u, \quad (9.8)$$

with

$$\omega_j(z, \lambda) = \sum_{v=1}^p \sum_{s=0}^p \lambda^{-s} \binom{p-v}{s} \theta_v D^s \bar{\gamma}_{\mu-v-s}.$$

A comparison of this with the earlier result (6.6) shows that

$$L^*(u) = L(u) - \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \{ \varepsilon_j(z, \lambda) + \omega_j(z, \lambda) \} D^{n-j} u. \quad (9.9)$$

The equation (9.1), whose solutions are completely known, thus has coefficients which differ from those of the given equation (2.1) only by terms that are of at least the r^{th} degree in $1/\lambda$. It is, therefore, by definition, a related equation.

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