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with each $\sigma_{j,v}^{(0)}(z)$ analytic, and $\sigma_{j,r}^{(0)}(z, \lambda)$ bounded. We shall show that the elements $\alpha_{j,v}^{(0)}(z)$ in (6.7) may be so specified as to yield

$$\sigma_{j,v}^{(0)}(z) \equiv \begin{cases} 1 & \text{when } (j, v) = (1, 0), \quad v = 0, 1, 2, \dots, (r-1) \dots \\ 0 & \text{when } (j, v) \neq (1, 0). \end{cases} \quad (6.14)$$

The effect of this will be to give the formula (6.11) the form

$$m^*(\eta_i) = \lambda^q \left\{ v_i(z, \lambda) + \frac{1}{\lambda^r} \sum_{j=1}^p \lambda^{1-j} \sigma_{j,r}^{(0)}(z, \lambda) D^{j-1} v_i \right\}. \quad (6.15)$$

7. ANOTHER DETERMINATION OF COEFFICIENTS.

The dependence of the functions (6.12) upon the unspecified ones $\alpha_{j,v}^{(0)}(z)$ of (6.7) is advantageously set forth in terms of vector-matrix notation. To this end, let a column vector with the components $\varphi_i, i = 1, 2, \dots, p$, be denoted by (φ) and let the vector whose components are the terms in $1/\lambda^v$ of (φ) , namely with the components $\varphi_{i,v} i = 1, 2, \dots, p$, be denoted by $(\varphi)_v$. Also let H designate the square matrix

$$H = \begin{bmatrix} \lambda^{-1}D & 0 & 0 & - & - & -\bar{\beta}_p \\ 1 & \lambda^{-1}D & 0 & - & - & -\bar{\beta}_{p-1} \\ 0 & 1 & \lambda^{-1}D & - & - & -\bar{\beta}_{p-2} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ 0 & 0 & - & - & - & -\bar{\beta}_1 + \lambda^{-1}D \end{bmatrix} \quad (7.1)$$

the elements of which are in part functions of z and λ , and in part the indicated differential operator. Again let H_v designate the matrix that is obtainable from (7.1) by replacing its elements by their terms in $1/\lambda^v$. The relations (6.10) are then seen at once to take the form

$$(\alpha^{(k)}) = H(\alpha^{(k-1)}).$$

With iteration defined in the manner

$$H^{[k]}(\varphi) = H(H^{[k-1]}(\varphi)), \quad H^{[1]} = H, \quad (7.2)$$

and with $H^{[0]}$ signifying the unit matrix, it is then easily seen that

$$(\alpha^{(k)}) = H^{[k]}(\alpha^{(0)}). \quad (7.3)$$

The relation (6.12) may thus be written in the form

$$(\sigma^{(0)}) = J(\alpha^{(0)}), \quad (7.4)$$

with J standing for the matrix

$$J = \sum_{k=0}^q \bar{\gamma}_k H^{[q-k]}. \quad (7.5)$$

The evaluations

$$(\sigma^{(0)})_v = \sum_{j=0}^v J_j (\alpha^{(0)})_{v-j},$$

$$J_j = \sum_{k=0}^q \sum_{i=0}^j \bar{\gamma}_{k,i} H_{j-i}^{[q-k]},$$

evidently combine to yield the formula

$$(\sigma^{(0)})_v = \sum_{k=0}^q \sum_{j=0}^v \sum_{i=0}^j \bar{\gamma}_{k,i} H_{j-i}^{[q-k]} (\alpha^{(0)})_{v-j}. \quad (7.6)$$

In connection with this, certain observations are apropos. To begin with, the index value $j = 0$ implies $i = 0$, whereas by (6.5) and (4.1), $\bar{\gamma}_{k,0} = c_k(z)$. Further when $i = j$ the matrix $H_{j-i}^{[q-k]}$ reduces to precisely $K^{q-k}(z)$, with $K(z)$ as given in (3.3). On the basis of these facts the equation (7.6) may be arranged into the form

$$\sum_{k=0}^q c_k(z) K^{q-k}(z) (\alpha^{(0)})_v = (\sigma^{(0)})_v - \sum_{k=0}^q \sum_{j=1}^v \sum_{i=0}^j \bar{\gamma}_{k,i} H_{j-i}^{[q-k]} (\alpha^{(0)})_{v-j} \quad (7.7)$$

This is a vector equation for $(\alpha^{(0)})_v$, which we shall consider for successive values of v , assuming that the values (6.14) have been assigned.

When $v = 0$, the triple sum on the right of the equality in (7.7) vanishes, and the right-hand member is, therefore, the vector $(\sigma^{(0)})_0$ whose first component is 1 and whose other components are 0. The equation is therefore a non-homogeneous

one, and accordingly admits of an analytic solution for $(a^{(0)})_0$ provided the matrix multiplier of this vector on the left is non-singular. This condition is assured by the relation (3. 4).

Now we may proceed by induction. Assuming that the vectors $(a^{(0)})_j$ for $j = 1, 2, \dots, (v-1)$, have been determined and are analytic, the right-hand member of the equation (7. 7) is known. As in the case $v = 0$, so now, the equation is analytically solvable. The solutions for the successive values $v = 0, 1, 2, \dots, (r-1)$, yield the coefficients (6. 7) for which the functions $\eta_i(z, \lambda)$, as given by the formulas (6. 8), fulfill the relations (6. 5).

8. ON LINEAR INDEPENDENCE.

With the functions $a_j^{(0)}(z, \lambda)$ now at hand, we have at our disposal the n known functions $y_j(z, \lambda)$, $j = 1, 2, \dots, q$, which are the solutions of the differential equation (6. 3), and $\eta_i(z, \lambda)$, $i = 1, 2, \dots, p$, which are given by the formulas (6. 8). We shall show that these functions are linearly independent.

Let the Wronskians of the entire set and of the respective sub-sets be denoted respectively by $W_n, W_q(y)$ and $W_p(\eta)$. If the usual form

$$W_n = \begin{bmatrix} y_1 & - & - & - & y_q & \eta_1 & - & - & - & \eta_p \\ Dy_1 & - & - & - & Dy_q & D\eta_1 & - & - & - & D\eta_p \\ - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ D^{n-1}y_1 & - & - & - & D^{n-1}y_q & D^{n-1}\eta_1 & - & - & - & D^{n-1}\eta_p \end{bmatrix} \quad (8. 1)$$

is modified by adding to each of the last p rows suitable multiples of the preceding ones, the formula can be made to appear thus

$$= \begin{bmatrix} y_1 & - & - & - & y_q & \eta_1 & - & - & - & \eta_p \\ Dy_1 & - & - & - & Dy_q & D\eta_1 & - & - & - & D\eta_p \\ - & - & - & - & - & - & - & - & - & - \\ D^{q-1}y_1 & - & - & - & D^{q-1}y_q & D^{q-1}\eta_1 & - & - & - & D^{q-1}\eta_p \\ m^*(y_1) & - & - & - & m^*(y_q) & m^*(\eta_1) & - & - & - & m^*(\eta_p) \\ Dm^*(y_1) & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ D^{p-1}m^*(y_1) & - & - & - & D^{p-1}m^*(y_q) & D^{p-1}m^*(\eta_1) & - & - & - & D^{p-1}m^*(\eta_p) \end{bmatrix} \quad (8. 2)$$