6. A SET OF FUNCTIONS $\eta_i(z,\lambda)$.

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6. A set of functions $\eta_i(z, \lambda)$.

It is the immediate purpose of this paper to show that under the hypotheses that have been made the construction of a related equation for the differential equation (2. 1) is referable to such constructions for equations of lower order. These latter will be precisely the differential equations (5. 6). In proceeding, we shall therefore suppose that the related equations

$$l^*(v) = 0, \tag{6.1}$$

with

$$l^*(v) \equiv \sum_{j=0}^{p} \lambda^j \bar{\beta}_j(z, \lambda) D^{p-j} v, \tag{6.2}$$

and

$$m^*(y) = 0, \tag{6.3}$$

with

$$m^*(y) \equiv \sum_{i=0}^{q} \lambda^i \bar{\gamma}_i(z, \lambda) D^{q-i} y \tag{6.4}$$

have been brought to hand. Their coefficients are of the forms

$$\bar{\beta}_j(z, \lambda) = \beta_j(z, \lambda) + \frac{1}{\lambda^r} \bar{\beta}_{j,r}(z, \lambda),$$

$$\bar{\gamma}_j(z, \lambda) = \gamma_j(z, \lambda) + \frac{1}{\lambda^r} \bar{\gamma}_{j,r}(z, \lambda), \tag{6.5}$$

the functions $\bar{\beta}_{i,r}$ and $\bar{\gamma}_{j,r}$ being bounded over the domains of z and λ. Since terms of a degree greater than the $(r-1)^{th}$ in $1/\lambda$ were never involved in the derivation of the relation (5. 5), it is clear that also

$$L(u) = l^*(m^*(u)) + \frac{1}{\lambda^r} \sum_{j=1}^{n} \lambda^j \epsilon_j(z, \lambda) D^{n-j} u, \tag{6.6}$$

with certain bounded coefficients ε_j.

As for all related equations, by definition, the solutions of the equations (6. 1) and (6. 3) are known. We shall denote complete sets for these solutions respectively by $v_i(z, \lambda)$, $i = 1, 2, \dots\ p$; and $y_j(z, \lambda)$, $j = 1, 2, \dots, q$. Our method, as will be seen, requires a modification of one of these solution sets. That is the purpose of the replacement of the functions $v_i(z, \lambda)$ by corresponding ones $\eta_i(z, \lambda)$ to which we proceed.

Let the functions $a_j^{(o)}(z, \lambda)$ be polynomials of the degree $(r-1)$ in $1/\lambda$, namely

$$\alpha_j^{(0)}(z, \lambda) = \sum_{\nu=0}^{r-1} \frac{\alpha_{j,\nu}^{(0)}(z)}{\lambda^\nu}, \tag{6.7}$$

with coefficients $a_{j,\nu}^{(0)}(z)$ that are analytic but, for the time being, unspecified. Then consider the formulas

$$\eta_i(z, \lambda) = \sum_{j=1}^{p} \lambda^{1-j} \alpha_j^{(0)}(z, \lambda) D^{j-1} v_i(z, \lambda), \quad i = 1, 2, \dots, p. \tag{6.8}$$

The differentiation of this, and the subsequent elimination of $D^p v_i$ by use of the equation (6. 1), yields a corresponding formula for $D\eta_i$, and repetitions of the procedure yield more generally that

$$D^k \eta_i = \lambda^k \sum_{j=1}^{p} \lambda^{i-j} \alpha_j^{(k)}(z, \lambda) D^{j-1} v_i(z, \lambda), \quad k = 0, 1, 2, \dots, q. \tag{6.9}$$

In this the coefficients are recursively given by the relation

$$\alpha_j^{(k)} = \alpha_{j-1}^{(k-1)} - \bar{\beta}_{p+1-j}\, \alpha_p^{(k-1)} + \frac{1}{\lambda} D\alpha_j^{(k-1)}. \tag{6.10}$$

The relations (6. 9) yield the formula

$$m^*(\eta_i) = \lambda^q \sum_{j=1}^{p} \lambda^{1-j} \sigma_j^{(0)}(z, \lambda) D^{j-1} v_i. \tag{6.11}$$

with

$$\sigma_j^{(0)}(z, \lambda) = \sum_{k=0}^{q} \gamma_k\, \alpha_j^{(q-k)}. \tag{6.12}$$

These latter coefficients $\sigma_j^{(0)}(z, \lambda)$ are evidently expressible in powers of $1/\lambda$, and are therefore of the form

$$\sigma_j^{(0)}(z, \lambda) \sum_{\nu=0}^{r-1} \frac{\sigma_{j,\nu}^{(0)}(z)}{\lambda^\nu} + \frac{\sigma_{j,r}^{(0)}(z, \lambda)}{\lambda^r} \tag{6.13}$$

with each $\sigma_{j,\nu}^{(0)}(z)$ analytic, and $\sigma_{j,r}^{(0)}(z, \lambda)$ bounded. We shall show that the elements $\alpha_{j,\nu}^{(0)}(z)$ in (6. 7) may be so specified as to yield

$$\sigma_{j,\nu}^{(0)}(z) \equiv \begin{cases} 1 \text{ when } (j,\nu) = (1,0), \quad \nu = 0, 1, 2, \ldots, (r-1) \\ 0 \text{ when } (j,\nu) \neq (1,0). \end{cases} \tag{6, 14}$$

The effect of this will be to give the formula (6. 11) the form

$$m^*(\eta_i) = \lambda^q \left\{ v_i(z, \lambda) + \frac{1}{\lambda^r} \sum_{j=1}^{p} \lambda^{1-j} \sigma_{j,r}^{(0)}(z, \lambda) D^{j-1} v_i \right\}. \tag{6. 15}$$

7. Another determination of coefficients.

The dependence of the functions (6. 12) upon the unspecified ones $\alpha_{j,\nu}^{(0)}(z)$ of (6. 7) is advantageously set forth in terms of vector-matrix notation. To this end, let a column vector with the components φ_i, $i = 1, 2, \ldots, p$, be denoted by (φ) and let the vector whose components are the terms in $1/\lambda^\nu$ of (φ), namely with the components $\varphi_{i,\nu}$ $i = 1, 2, \ldots, p$, be denoted by $(\varphi)_\nu$. Also let H designate the square matrix

$$H = \begin{bmatrix} \lambda^{-1}D & 0 & 0 & - & - & -\bar\beta_p \\ 1 & \lambda^{-1}D & 0 & - & - & -\bar\beta_{p-1} \\ 0 & 1 & \lambda^{-1}D & - & - & -\bar\beta_{p-2} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ 0 & 0 & - & - & - & -\bar\beta_1 + \lambda^{-1}D \end{bmatrix} \tag{7. 1}$$

the elements of which are in part functions of z and λ, and in part the indicated differential operator. Again let H_ν designate the matrix that is obtainable from (7. 1) by replacing its elements by their terms in $1/\lambda^\nu$. The relations (6. 10) are then seen at once to take the form

$$(\alpha^{(k)}) = H(\alpha^{(k-1)}).$$

With iteration defined in the manner

$$H^{[k]}(\varphi) = H(H^{[k-1]}(\varphi)), \quad H^{[1]} = H, \tag{7. 2}$$