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Artikel: ON THE CONSTRUCTION OF RELATED EQUATIONS FOR THE ASYMPTOTIC THEORY OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS ABOUT A TURNING POINT

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Kapitel: 5. A DETERMINATION OF UNSPECIFIED COEFFICIENTS.

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$$l(m(y)) = \sum_{i=0}^{n} \lambda^i \Psi_i(z, \lambda) D^{n-i} y, \tag{4.4}$$

with

$$\Psi_i(z, \lambda) = \sum_{j=0}^{p} \sum_{s=0}^{p-j} \lambda^{-s} \binom{p-j}{s} \beta_j D^s \gamma_{i-j-s} \,. \tag{4.5}$$

The functions $\Psi_i(z, \lambda)$, inasmuch as they are combinations of those given in (4. 1), are polynomials in $1/\lambda$. We may therefore write them in the form

$$\Psi_i(z, \lambda) = \sum_{\mu=0}^{r-1} \frac{\psi_{i,\mu}(z)}{\lambda^\mu} + \frac{\psi_{i,r}(z, \lambda)}{\lambda^r} \,. \tag{4.6}$$

A comparison of the terms in like powers of $1/\lambda$ in the relations (4. 5) and (4. 6) yields formulas for the functions $\Psi_{i,\mu}(z)$. Those for which $\mu = 0$ are particularly easy to obtain. On setting $s = 0$ in (4. 5), and replacing β_j and γ_{i-j} by their leading terms b_j and c_{i-j}, we find that

$$\psi_{i,0}(z) = \sum_{j=0}^{p} b_j(z) c_{i-j}(z) \,.$$

Recourse to the relation (2. 8) thus shows that

$$\psi_{i,0}(z) = p_{i,0}(z), \quad i = 1, 2, \ldots, n \,. \tag{4.7}$$

At least to the extent of the leading terms of their coefficients, the forms (2. 2) and (4. 4) are, therefore, the same.

5. A DETERMINATION OF UNSPECIFIED COEFFICIENTS.

We propose now to deduce a formula for the general coefficient $\psi_{i,\mu}(z)$ in (4. 6) by selecting the multiplier of the appropriate power of $1/\lambda$ from the formula (4. 5). To begin with, it follows from the relations (4. 1) that

$$\beta_j D^s \gamma_{i-j-s} = \sum_{\mu=0}^{2r-2} \sum_{k=0}^{\mu} \lambda^{-\mu} \beta_{j,k} D^s \gamma_{i-j-s,\mu-k}$$

By virtue of this, the relation (4. 5) may be more precisely written as

$$\psi_i(z, \lambda) = \sum_{j=0}^{p} \sum_{s=0}^{p-j} \sum_{\mu=0}^{2r-2} \sum_{k=0}^{\mu} \lambda^{-s-\mu} \binom{p-j}{s} \beta_{j,k} D^s \gamma_{i-j-s,\mu-k} \cdot$$

By the use of $\mu+s$ as a variable of summation in place of μ, this is, however, seen to take the form (4. 6) with

$$\psi_{i,\mu}(z) = \sum_{j=0}^{p} \sum_{s=0}^{p-j} \sum_{k=0}^{\mu=s} \binom{p-j}{s} \beta_{j,k} D^s \gamma_{i-j-s,\mu-s-k} \cdot \qquad (5.1)$$

An inspection of this result reveals an important fact, namely, that the functions $\psi_{i,\mu}(z)$, with any specific μ, do not depend at all upon any of the elements $\beta_{j,i}(z)$, $\gamma_{j,i}(z)$ for which $i > \mu$. Moreover these elements with $i = \mu$ are involved in them precisely to the respective extent

$$\sum_{j=0}^{p} \{ \beta_{j,\mu} \gamma_{i-j,0} + \beta_{j,0} \gamma_{i-j,\mu} \},$$

namely, on dropping the terms to which the value zero must be assigned, reversing one of the summations, and recalling that $\beta_{j,0} = b_j$ and $\gamma_{j,0} = c_j$, to the extent

$$\sum_{j=0}^{i} \{ b_{i-j} \gamma_{j,\mu} + c_{i-j} \beta_{j,\mu} \}.$$

The formulas (5. 1) therefore have the form

$$\psi_{i,\mu}(z) = \sum_{j=l}^{i} \{ b_{i-j} \gamma_{j,\mu} + c_{i-j} \beta_{j,\mu} \} + \varphi_{i,\mu}(z), \qquad (5.2)$$

with $\varphi_{i,\mu}(z)$ denoting a function which is contructed of the elements $\beta_{j,i}$ and $\gamma_{j,i}$ in which $i < \mu$.

We recall now that the elements $\beta_{j,i}(z)$, $\gamma_{j,i}(z)$ with $i \geqq 1$ were left unspecified, except that they be analytic, and inquire whether they may be so specified as to make the formulas (5. 2) yield assigned functions. The particular assignment envisaged is

$$\psi_{i,\mu}(z) = p_{i,\mu}(z), \ i = 1, 2, \ldots, n; \ \mu = 1, 2, \ldots, (r-1). \qquad (5.3)$$

This question is, in other terms, whether the equations

$$\sum_{j=1}^{i} \{ b_{i,j}\gamma_{j,\mu} + c_{i-j}\beta_{j,\mu} \} = p_{i,\mu}(z) - \varphi_{i,\mu}(z), \quad \begin{array}{l} i = 1, 2, \ldots, n; \\ \mu = 1, 2, \ldots, (r-1), \end{array} \tag{5.4}$$

can be fulfilled by choice of the functions $\beta_j(z, \lambda)$, $\gamma_j(z, \lambda)$ of (4. 1).

Consider first the case in which $\mu = 1$. In this case the right-hand members of the equations are known, since the functions $\varphi_{i,1}(z)$ are made up of the known elements $b_j(z)$, $c_j(z)$. The equations therefore comprise a linear non-homogeneous systems in the " unknowns " $\gamma_{1,1} \cdots \gamma_{q,1}$, $\beta_{1,1}, \cdots \beta_{p,1}$, and the determinant of this system is seen to be $\Delta(z)$, the determinant (3. 2), written with rows and columns interchanged. Since this is nowhere zero in the z-region, by (3. 1), the system is analytically solvable, and by the solution the equations (5. 3) for $\mu = 1$ are assured.

We proceed now by induction. Assuming that the elements $\beta_{j,i}$, $\gamma_{j,i}$ have been determined for $i = 1, 2, \ldots, (\mu - 1)$, we consider the system (5. 4). The right-hand members of the equations are known, and the determinant of the system is $\Delta(z)$. The system is, therefore, analytically solvable for $\gamma_{1,\mu}, \cdots \gamma_{q,\mu}$, $\beta_{1,\mu}, \cdots \beta_{p,\mu}$, for successive values of μ. By these solutions the equations (5. 3) are fulfilled, and now, from a comparison of the formula (2. 2) with (4. 4), and of (2. 3) with (4. 6) and (5. 3), we see that

$$L(u) = l(m(u)) + \frac{1}{\lambda^r} \sum_{j=1}^{n} \lambda^j \{ p_{j,r}(z, \lambda) - \psi_{j,r}(z, \lambda) \} D^{n-j} u. \tag{5.5}$$

The differential operators (4. 2), and therewith the differential equations

$$\begin{aligned} l(v) &= 0, \\ m(y) &= 0, \end{aligned} \tag{5.6}$$

are now completely specific.