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$$l(m(y)) = \sum_{i=0}^n \lambda^i \Psi_i(z, \lambda) D^{n-i} y, \quad (4.4)$$

with

$$\Psi_i(z, \lambda) = \sum_{j=0}^p \sum_{s=0}^{p-j} \lambda^{-s} \binom{p-j}{s} \beta_j D^s \gamma_{i-j-s}. \quad (4.5)$$

The functions $\Psi_i(z, \lambda)$, inasmuch as they are combinations of those given in (4.1), are polynomials in $1/\lambda$. We may therefore write them in the form

$$\Psi_i(z, \lambda) = \sum_{\mu=0}^{r-1} \frac{\psi_{i,\mu}(z)}{\lambda^\mu} + \frac{\psi_{i,r}(z, \lambda)}{\lambda^r}. \quad (4.6)$$

A comparison of the terms in like powers of $1/\lambda$ in the relations (4.5) and (4.6) yields formulas for the functions $\psi_{i,\mu}(z)$. Those for which $\mu = 0$ are particularly easy to obtain. On setting $s = 0$ in (4.5), and replacing β_j and γ_{i-j} by their leading terms b_j and c_{i-j} , we find that

$$\psi_{i,0}(z) = \sum_{j=0}^p b_j(z) c_{i-j}(z).$$

Recourse to the relation (2.8) thus shows that

$$\psi_{i,0}(z) = p_{i,0}(z), \quad i = 1, 2, \dots, n. \quad (4.7)$$

At least to the extent of the leading terms of their coefficients, the forms (2.2) and (4.4) are, therefore, the same.

5. A DETERMINATION OF UNSPECIFIED COEFFICIENTS.

We propose now to deduce a formula for the general coefficient $\psi_{i,\mu}(z)$ in (4.6) by selecting the multiplier of the appropriate power of $1/\lambda$ from the formula (4.5). To begin with, it follows from the relations (4.1) that

$$\beta_j D^s \gamma_{i-j-s} = \sum_{\mu=0}^{2r-2} \sum_{k=0}^{\mu} \lambda^{-\mu} \beta_{j,k} D^s \gamma_{i-j-s, \mu-k}$$

By virtue of this, the relation (4.5) may be more precisely written as

$$\psi_i(z, \lambda) = \sum_{j=0}^p \sum_{s=0}^{p-j} \sum_{\mu=0}^{2r-2} \sum_{k=0}^{\mu} \lambda^{-s-\mu} \binom{p-j}{s} \beta_{j,k} D^s \gamma_{i-j-s, \mu-k}.$$

By the use of $\mu+s$ as a variable of summation in place of μ , this is, however, seen to take the form (4.6) with

$$\psi_{i,\mu}(z) = \sum_{j=0}^p \sum_{s=0}^{p-j} \sum_{k=0}^{\mu} \binom{p-j}{s} \beta_{j,k} D^s \gamma_{i-j-s, \mu-s-k}. \quad (5.1)$$

An inspection of this result reveals an important fact, namely, that the functions $\psi_{i,\mu}(z)$, with any specific μ , do not depend at all upon any of the elements $\beta_{j,i}(z)$, $\gamma_{j,i}(z)$ for which $i > \mu$. Moreover these elements with $i = \mu$ are involved in them precisely to the respective extent

$$\sum_{j=0}^p \{ \beta_{j,\mu} \gamma_{i-j,0} + \beta_{j,0} \gamma_{i-j,\mu} \},$$

namely, on dropping the terms to which the value zero must be assigned, reversing one of the summations, and recalling that $\beta_{j,0} = b_j$ and $\gamma_{j,0} = c_j$, to the extent

$$\sum_{j=0}^i \{ b_{i-j} \gamma_{j,\mu} + c_{i-j} \beta_{j,\mu} \}.$$

The formulas (5.1) therefore have the form

$$\psi_{i,\mu}(z) = \sum_{j=l}^i \{ b_{i-j} \gamma_{j,\mu} + c_{i-j} \beta_{j,\mu} \} + \varphi_{i,\mu}(z), \quad (5.2)$$

with $\varphi_{i,\mu}(z)$ denoting a function which is constructed of the elements $\beta_{j,i}$ and $\gamma_{j,i}$ in which $i < \mu$.

We recall now that the elements $\beta_{j,i}(z)$, $\gamma_{j,i}(z)$ with $i \geq 1$ were left unspecified, except that they be analytic, and inquire whether they may be so specified as to make the formulas (5.2) yield assigned functions. The particular assignment envisaged is

$$\psi_{i,\mu}(z) = p_{i,\mu}(z), \quad i = 1, 2, \dots, n; \quad \mu = 1, 2, \dots, (r-1). \quad (5.3)$$

This question is, in other terms, whether the equations

$$\sum_{j=1}^i \{ b_{i,j} \gamma_{j,\mu} + c_{i-j} \beta_{j,\mu} \} = p_{i,\mu}(z) - \varphi_{i,\mu}(z), \quad \begin{aligned} i &= 1, 2, \dots, n; \\ \mu &= 1, 2, \dots, (r-1), \end{aligned} \quad (5.4)$$

can be fulfilled by choice of the functions $\beta_j(z, \lambda)$, $\gamma_j(z, \lambda)$ of (4.1).

Consider first the case in which $\mu = 1$. In this case the right-hand members of the equations are known, since the functions $\varphi_{i,1}(z)$ are made up of the known elements $b_j(z)$, $c_j(z)$. The equations therefore comprise a linear non-homogeneous systems in the "unknowns" $\gamma_{1,1} \dots \gamma_{q,1}$, $\beta_{1,1}, \dots \beta_{p,1}$, and the determinant of this system is seen to be $\Delta(z)$, the determinant (3.2), written with rows and columns interchanged. Since this is nowhere zero in the z -region, by (3.1), the system is analytically solvable, and by the solution the equations (5.3) for $\mu = 1$ are assured.

We proceed now by induction. Assuming that the elements $\beta_{j,i}$, $\gamma_{j,i}$ have been determined for $i = 1, 2, \dots, (\mu-1)$, we consider the system (5.4). The right-hand members of the equations are known, and the determinant of the system is $\Delta(z)$. The system is, therefore, analytically solvable for $\gamma_{1,\mu}, \dots \gamma_{q,\mu}$, $\beta_{1,\mu}, \dots \beta_{p,\mu}$, for successive values of μ . By these solutions the equations (5.3) are fulfilled, and now, from a comparison of the formula (2.2) with (4.4), and of (2.3) with (4.6) and (5.3), we see that

$$L(u) = l(m(u)) + \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \{ p_{j,r}(z, \lambda) - \psi_{j,r}(z, \lambda) \} D^{n-j} u. \quad (5.5)$$

The differential operators (4.2), and therewith the differential equations

$$l(v) = 0, \quad (5.6)$$

$$m(y) = 0,$$

are now completely specific.