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<b>Autor:</b>	Langer, Rudolph E.
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The relations (2.5), (2.6) and (2.7) imply that  $p+q = n$ . We shall suppose the notation to be so assigned as to yield  $p \leq q$ . It follows then also that

$$\sum_{i=0}^k b_i(z) c_{k-i}(z) = p_{k,0}(z), \quad k = 0, 1, 2, \dots, n, \quad (2.8)$$

it being understood that the value 0 is to be assigned to any symbol  $b_i(z)$  or  $c_{k-i}(z)$  which, by virtue of its subscript, is not present in the formulas (2.7).

A change of dependent variable can be applied to the equation (1.1) to give it a form for which the coefficient  $b_1(z)$  is identically zero. Considerable simplifications of the formulas result therefrom. We shall not resort to that normalization, however, refraining from it in order to keep the roles of the factors  $B(\chi)$  and  $\Gamma(\chi)$  interchangeable.

### 3. THE RESULTANT.

It is a hypothesis that the two polynomials (2.7) are relatively prime over the  $z$ -region. If we denote their resultant by  $\Delta(z)$ , we have accordingly

$$\Delta(z) \neq 0, \quad (3.1)$$

with

$$\Delta(z) \equiv \begin{vmatrix} 1 & b_1 & b_2 & \cdots & b_p & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & b_1 & b_2 & \cdots & b_p & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots \\ 0 & \cdots & 0 \\ 1 & c_1 & c_2 & \cdots & \cdots & \cdots & c_q & 0 & \cdots & 0 \\ 0 & 1 & c_1 & \cdots & \cdots & \cdots & c_q & 0 & \cdots & 0 \\ \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & c_1 & c_2 & \cdots & \cdots & c_q \end{vmatrix} \quad (3.2)$$

We shall find use for the relation (3.2). However, for future use we find it convenient to formulate the relative primary of

the factors  $B(\chi)$  and  $\Gamma(\chi)$  also in an alternative form. This is done as follows:

Let  $K$  designate the familiar matrix

$$K(z) = \begin{bmatrix} 0 & 0 & - & - & - & - & - & -b_p \\ 1 & 0 & - & - & - & - & - & -b_{p-1} \\ 0 & 1 & 0 & - & - & - & - & -b_{p-2} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & - & - & - & - & 1 & -b_1 & \end{bmatrix} \quad (3.3)$$

The eigen-values of this are the roots  $x_i$ ,  $i = 1, 2, \dots, p$  of the equation  $B(x) = 0$ . If we designate by  $\xi_i$  an eigen-vector corresponding to  $x_i$  we have

$$K^h \xi_i = x_i^h \xi_i, \quad h = 1, 2, \dots, q,$$

and accordingly

$$\Gamma(K) \xi_i = \Gamma(x_i) \xi_i.$$

Thus  $\Gamma(x_i)$  is an eigenvalue of the matrix  $\Gamma(K)$ , and, since the product of the eigenvalues is the determinant of the matrix, we have

$$|\Gamma(K)| = \prod_{i=1}^p \Gamma(x_i).$$

Observing that no factor on the right is zero, and giving to  $\Gamma(K)$  its explicit form, we conclude with the result

$$|\sum_{j=0}^q c_j(z) K^{q-j}(z)| \neq 0 \quad (3.4)$$

#### 4. TWO DIFFERENTIAL OPERATORS OF THE ORDERS $p$ AND $q$ .

Let the functions  $\beta_j(z, \lambda)$  and  $\gamma_i(z, \lambda)$  be taken to be polynomials of the degree  $(r - 1)$  in  $1/\lambda$ , thus

$$\begin{aligned} \beta_j(z, \lambda) &= \sum_{v=0}^{r-1} \frac{\beta_{j,v}(z)}{\lambda^v}, \quad \beta_{j,0}(z) \equiv b_j(z); \quad j = 1, 2, \dots, p, \\ \gamma_i(z, \lambda) &= \sum_{v=0}^{r-1} \frac{\gamma_{i,v}(z)}{\lambda^v}, \quad \gamma_{i,0}(z) \equiv c_i(z); \quad i = 1, 2, \dots, q. \end{aligned} \quad (4.1)$$