Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 8 (1962)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE CONSTRUCTION OF RELATED EQUATIONS FOR THE ASYMPTOTIC THEORY OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS ABOUT A TURNING POINT

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Kapitel: 2. The hypotheses.

DOI: https://doi.org/10.5169/seals-37963

rapidly with the order of the differential equation, and therefore any means for referring the problem from a given equation to ones of lower order are treasurable. This brings us to the point. In an earlier proper [5] I have shown that if, in a given region, p of the auxiliary roots of a differential equation (1. 1) are simple, the construction of a related equation is referrable to such a construction for an equation of the lower order $(n-p)$. The present paper goes materially further. It shows that if the auxiliary polynomial in χ factors into relatively prime factors of the degrees p and q, the coefficients of which are analytic, the construction of a related equation is referrable to such constructions for differential equations of the lower orders p and q. It will be seen at once, by iteration, that if the auxiliary polynomial has analytic factors of the degrees $p_1, p_2, \dots p_k$, the construction of a related equation is referrable to such constructions for differential equations of the orders $p_1, p_2, \cdots p_k$. The equation (1. 3) serves again as an example. Its auxiliary equation may be written

$$(\chi^2+z)(\chi^2-2\chi+1-z) = 0 .$$

A related equation for it can be constructed, because that can be done for the two equations of lower order

$$\frac{d^2 v}{dz^2} + \lambda^2\left(z-\frac{1}{\lambda}\right)v = 0 ,$$

$$\frac{d^2 y}{dz^2} - 2\lambda\frac{dy}{dz} + \lambda^2(1-z)y = 0 .$$

2. The hypotheses.

The given differential equation (1. 1) may be conveniently denoted by

$$L(u) = 0 , \qquad (2.1)$$

with

$$L(u) \equiv \sum_{j=0}^{n} \lambda^j p_j(z,\lambda) D^{n-j} u , \quad p_0 \equiv 1 , \qquad (2.2)$$

the symbol $D^k f$ signifying $d^k f/dz^k$. It is an overall objective of this discussion to describe the construction of another differential equation subject to the specification, among others, that its coefficients be the same as those of (2. 2) to the extent of all terms which, in terms of $1/\lambda$, are of a degree less than a certain arbitrarily prescribed one. We shall designate this prescribed degree as the r^{th}. The domains of the variable z and the parameter λ shall be respectively a closed region of the complex z-plane and any portion of the complex λ-plane in which $|\lambda|$ is bounded below by a positive bound but is unbounded above, and in which the coefficients of (2. 2) have the forms

$$p_j(z, \lambda) = \sum_{\mu=0}^{r-1} \frac{p_{j,\mu}(z)}{\lambda^\mu} + \frac{p_{j,r}(z, \lambda)}{\lambda^r}, \quad j = 1, 2, \ldots, n, \qquad (2.3)$$

with each $p_{j,\mu}(z)$, $\mu < r$ analytic, and $p_{j,r}(z, \lambda)$ bounded.

The auxiliary equation (1. 2) is accordingly

$$P(\chi) = 0, \qquad (2.4)$$

with

$$P(\chi) = \sum_{j=0}^{n} p_{j,0}(z)\, \chi^{n-j}. \qquad (2.5)$$

It shall be a hypothesis that in the z-region under consideration the polynomial $P(\chi)$ factors, thus

$$P(\chi) = B(\chi)\, \Gamma(\chi), \qquad (2.6)$$

the factors

$$B(\chi) = \sum_{j=0}^{p} b_j(z)\, \chi^{p-j}, \quad b_0 \equiv 1,$$
$$\Gamma(\chi) = \sum_{j=0}^{q} c_j(z)\, \chi_q^{q-j}, \quad c_0 \equiv 1, \qquad (2.7)$$

being relatively prime, and having analytic coefficients. The z-region shall contain no more than one turning-point, and if there is such a point it shall be taken to be the origin.

The relations (2. 5), (2. 6) and (2. 7) imply that $p+q=n$. We shall suppose the notation to be so assigned as to yield $p \leqq q$. It follows then also that

$$\sum_{i=0}^{k} b_i(z)\, c_{k-i}(z) = p_{k,0}(z), \quad k = 0,\ 1,\ 2,\ \ldots,\ n, \tag{2. 8}$$

it being understood that the value 0 is to be assigned to any symbol $b_i(z)$ or $c_{k-i}(z)$ which, by virtue of its subscript, is not present in the formulas (2. 7).

A change of dependent variable can be applied to the equation (1. 1) to give it a form for which the coefficient $b_1(z)$ is identically zero. Considerable simplifications of the formulas result therefrom. We shall not resort to that normalization, however, refraining from it in order to keep the roles of the factors $B(\chi)$ and $\Gamma(\chi)$ interchangeable.

3. The resultant.

It is a hypothesis that the two polynomials (2. 7) are relatively prime over the z-region. If we denote their resultant by $\varDelta(z)$, we have accordingly

$$\varDelta(z) \neq 0, \tag{3. 1}$$

with

$$\varDelta(z) \equiv \begin{vmatrix}
1 & b_1 & b_2 & - & - & - & - & b_p & 0 & - & - & - & - & - & 0 \\
0 & 1 & b_1 & b_2 & - & - & - & - & b_p & 0 & - & - & - & - & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & - & - & - & - & - & - & - & 0 & 1 & b_1 & - & - & - & b_p \\
1 & c_1 & c_2 & - & - & - & - & - & - & - & - & c_q & 0 & - & 0 \\
0 & 1 & c_1 & - & - & - & - & - & - & - & - & - & c_q & 0 & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & - & - & - & - & 0 & 1 & c_1 & c_2 & - & - & - & - & - & c_q
\end{vmatrix} \tag{3. 2}$$

We shall find use for the relation (3. 2). However, for future use we find it convenient to formulate the relative primary of