Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 8 (1962)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NEW FIXED POINT THEOREM FOR CONTINUOUS MAPS OF THE

CLOSED n-CELL

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**Kapitel:** 5. The Theorems

**DOI:** https://doi.org/10.5169/seals-37951

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Let u be the center of  $S^{n-1}$ . Since f has no fixed point, it is clear that we can choose d > 0 so small that a closed solid n-sphere  $H_d^n$  of radius d with center at  $\theta$  (u) is entirely in  $\eta^n$ , and  $H_d^n$  and its image f ( $H_d^n$ ) are contained in different half-spaces into which  $R^n$  is separated by some (n-1)-plane.

Now, let  $S^{n-1}$  undergo a deformation by uniform radial shrinking toward u till it reaches a position  $S_2^{n-1}$  whose image  $\sigma_2^{n-1}$  under  $\theta$  is contained in the interior of  $H_d^n$ . By means of  $\theta$ , there results a deformation of  $\sigma^{n-1}$  into  $\sigma_2^{n-1}$  which by means of the mapping f induces a deformation, on the direction sphere, of the (n-1)-cycle  $f^{n-1}$  resulting from f applied to  $\sigma_2^{n-1}$ .

Thus the turning index of  $\sigma^{n-1}$  under f equals the turning index of  $\sigma_2^{n-1}$  under f, which by Lemma 2 equals zero. Thus Lemma 4 is proved.

# 5. The Theorems

Theorem 1. Let  $\eta^n \subset \mathbb{R}^n$  be a closed n-cell and f a continuous mapping of  $\eta^n$  into  $\mathbb{R}^n$  such that f maps the boundary  $\sigma^{n-1}$  of  $\eta^n$  into  $\eta^n$ . Then f has at least one fixed point.

*Proof.* Assume no fixed points. Let, as in the case of Lemma 3,  $\eta^n$  and  $\sigma^{n-1}$  be respectively the images (under the homeomorphism  $\theta$ ) of the closed solid *n*-sphere  $E^n$  with boundary  $S^{n-1}$ , i.e.,  $\eta^n = \theta$  ( $E^n$ ) and  $\sigma^{n-1} = \theta$  ( $S^{n-1}$ ).

Let u be the center of  $S^{n-1}$ . Consider the mapping f' of  $\sigma^{n-1}$  which maps every point  $\sigma \in \sigma^{n-1}$  into the point  $\theta(u)$ . Since f' is the mapping which appears in the definition of the index of  $\theta(u)$  relative to  $\sigma^{n-1}$ , we see by Lemma 3 that the turning index of  $\sigma^{n-1}$  under f' is non-zero.

By hypothesis,  $f(\sigma) \in \eta^n$  for every  $\sigma \in \sigma^{n-1}$ . Hence we may deform  $f(\sigma^{n-1})$  as follows. As a parameter p varies from 0 to 1,

the point  $\sigma'$  moves in  $\eta^n$  along the path  $\theta[\overline{\theta^{-1}f(\sigma)}, u]$  starting from  $\sigma$  and ending at  $\theta(u)$ .

For p=1, the above deformation yields the mapping f'. Therefore, the (n-1)-cycle resulting from f applied to  $\sigma^{n-1}$  is homologous on the direction sphere (as a consequence of a deformation) to the (n-1)-cycle resulting from f' applied to

 $\sigma^{n-1}$ . Consequently, the turning index of  $\sigma^{n-1}$  under f equals the turning index of  $\sigma^{n-1}$  under f', and hence is not zero. But this contradicts Lemma 4. Thus, Theorem 1 is true.

Theorem 2. Let  $\eta^n \subset R^n$  be a closed n-cell with boundary  $\sigma^{n-1}$  and f a continuous map of  $\eta^n$  into  $R^n$  which leaves no point of  $\sigma^{n-1}$  fixed. If there exists an inner point e of  $\eta^n$  and an angle  $\alpha$  with  $0 \leq \alpha \leq \pi$ , such that for no point  $\sigma \in \sigma^{n-1}$  is  $\alpha$  an angle from the vector  $\overline{\sigma}$ ,  $\overline{f}(\sigma)$  to the vector  $\overline{e}$ ,  $\overline{\sigma}$  then f leaves at least one point fixed.

*Proof.* Suppose f leaves no point fixed. We shall show that under the hypotheses of Theorem 2, either

- i) for no point  $\sigma \in \sigma^{n-1}$  is the direction from  $\sigma$  to  $f(\sigma)$  opposite to that from e to  $\sigma$ ,
- ii) for no point  $\sigma \varepsilon \sigma^{n-1}$  is the direction from  $\sigma$  to  $f(\sigma)$  opposite to that from  $\sigma$  to e.

For, otherwise we would have points  $\sigma_1$  and  $\sigma_2 \in \sigma^{n-1}$  such that, as  $\sigma$  traverses a path from  $\sigma_1$  to  $\sigma_2$  on  $\sigma^{n-1}$ , the angle between  $\sigma$ ,  $f(\sigma)$  and  $\sigma$ , e would change continuously from 0 to  $\pi$ , hence assume the value  $\alpha$ , a contradiction.

- If i) holds, we apply Lemma 1 taking the mapping g of Lemma 1 as the mapping f, and as the mapping h, we take a mapping which makes correspond to each point  $\sigma \varepsilon \sigma^{n-1}$  the intersection of the half line starting at the point e and passing through the point  $\sigma$ , with an (n-1)-sphere  $V^{n-1}$  whose center is e and which is located completely outside of  $\sigma^{n-1}$ . We infer by Lemma 1 that the turning indices of  $\sigma^{n-1}$  under f and h are equal. Since the turning index of  $\sigma^{n-1}$  under h clearly equals the turning index of  $\sigma^{n-1}$  relative to  $V^{n-1}$ , we infer from Lemma 3 that the turning index of  $\sigma^{n-1}$  under f is non-zero.
- If ii) holds, again by Lemmas 1 and 3 the turning index of  $\sigma^{n-1}$  under f is non-zero. (Here, for the mapping g of Lemma 1, we again take the mapping f, and for the mapping h, we take a mapping which makes correspond to each point  $\sigma \varepsilon \sigma^{n-1}$  the intersection of the half line starting at the point e and passing through the point  $\sigma$ , with an (n-1)-sphere  $V^{n-1}$  whose center is e and which is located completely inside of  $\sigma^{n-1}$ ).

In short, the turning index of  $\sigma^{n-1}$  under the assumption of the absence of fixed points is non-zero, a fact which contradicts Lemma 4. Hence f has at least one fixed point, and Theorem 2 is proved.

COROLLARY 1. Let  $E^n$  be a closed solid n-sphere and f a continuous mapping of  $E^n$  into  $R^n$  such that f maps the boundary  $S^{n-1}$  of  $E^n$  into  $E^n$ . Then f has at least one fixed point.

*Proof.* If no point of  $S^{n-1}$  is fixed, then the hypotheses of Theorem 2 are seen to be satisfied with e at the center of the sphere  $E^n$  and  $\alpha = 0$ .

Clearly, Corollary 1 also follows immediately from Theorem 1. Proofs of this corollary also appear in the literature ([3], page 115).

COROLLARY 2. Let  $\eta^n \subset R^n$  be a closed n-cell with boundary  $\sigma^{n-1}$ , and f and g two continuous maps of  $\eta^n$  into  $R^n$  such that for no point  $\sigma \in \sigma^{n-1}$  is  $f(\sigma) = g(\sigma)$ . If there exists an inner point  $e(\sigma)$  of  $e(\sigma)$  and a constant angle  $e(\sigma)$ ,  $e(\sigma)$  is  $e(\sigma)$  and  $e(\sigma)$  and  $e(\sigma)$  and  $e(\sigma)$  between the vectors  $e(\sigma)$  and  $e(\sigma)$ , then there is a point  $e(\sigma)$  such that  $e(\sigma)$  is  $e(\sigma)$ .

*Proof.* Consider the map h of  $\eta^n$  into  $R^n$  such that for every point  $\eta \in \eta^n$  the vectors  $\overline{\eta}$ ,  $\overline{h}$  ( $\overline{\eta}$ ) and  $\overline{f}$  ( $\overline{\eta}$ ),  $\overline{g}$  ( $\overline{\eta}$ ) are equal. By Theorem 2, the map h has a fixed point  $\eta_0$ . Consequently,  $f(\eta_0) = g(\eta_0)$ .

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