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Let  $u$  be the center of  $S^{n-1}$ . Since  $f$  has no fixed point, it is clear that we can choose  $d > 0$  so small that a closed solid  $n$ -sphere  $H_d^n$  of radius  $d$  with center at  $\theta(u)$  is entirely in  $\eta^n$ , and  $H_d^n$  and its image  $f(H_d^n)$  are contained in different half-spaces into which  $R^n$  is separated by some  $(n-1)$ -plane.

Now, let  $S^{n-1}$  undergo a deformation by uniform radial shrinking toward  $u$  till it reaches a position  $S_2^{n-1}$  whose image  $\sigma_2^{n-1}$  under  $\theta$  is contained in the interior of  $H_d^n$ . By means of  $\theta$ , there results a deformation of  $\sigma^{n-1}$  into  $\sigma_2^{n-1}$  which by means of the mapping  $f$  induces a deformation, on the direction sphere, of the  $(n-1)$ -cycle  $f^{n-1}$  resulting from  $f$  applied to  $\sigma^{n-1}$  into the  $(n-1)$ -cycle  $f_2^{n-1}$  resulting from  $f$  applied to  $\sigma_2^{n-1}$ .

Thus the turning index of  $\sigma^{n-1}$  under  $f$  equals the turning index of  $\sigma_2^{n-1}$  under  $f$ , which by Lemma 2 equals zero. Thus Lemma 4 is proved.

## 5. THE THEOREMS

**THEOREM 1.** *Let  $\eta^n \subset R^n$  be a closed  $n$ -cell and  $f$  a continuous mapping of  $\eta^n$  into  $R^n$  such that  $f$  maps the boundary  $\sigma^{n-1}$  of  $\eta^n$  into  $\eta^n$ . Then  $f$  has at least one fixed point.*

*Proof.* Assume no fixed points. Let, as in the case of Lemma 3,  $\eta^n$  and  $\sigma^{n-1}$  be respectively the images (under the homeomorphism  $\theta$ ) of the closed solid  $n$ -sphere  $E^n$  with boundary  $S^{n-1}$ , i.e.,  $\eta^n = \theta(E^n)$  and  $\sigma^{n-1} = \theta(S^{n-1})$ .

Let  $u$  be the center of  $S^{n-1}$ . Consider the mapping  $f'$  of  $\sigma^{n-1}$  which maps every point  $\sigma \in \sigma^{n-1}$  into the point  $\theta(u)$ . Since  $f'$  is the mapping which appears in the definition of the index of  $\theta(u)$  relative to  $\sigma^{n-1}$ , we see by Lemma 3 that the turning index of  $\sigma^{n-1}$  under  $f'$  is non-zero.

By hypothesis,  $f(\sigma) \in \eta^n$  for every  $\sigma \in \sigma^{n-1}$ . Hence we may deform  $f(\sigma^{n-1})$  as follows. As a parameter  $p$  varies from 0 to 1, the point  $\sigma'$  moves in  $\eta^n$  along the path  $\overline{\theta[\theta^{-1}f(\sigma), u]}$  starting from  $\sigma$  and ending at  $\theta(u)$ .

For  $p = 1$ , the above deformation yields the mapping  $f'$ . Therefore, the  $(n-1)$ -cycle resulting from  $f$  applied to  $\sigma^{n-1}$  is homologous on the direction sphere (as a consequence of a deformation) to the  $(n-1)$ -cycle resulting from  $f'$  applied to

$\sigma^{n-1}$ . Consequently, the turning index of  $\sigma^{n-1}$  under  $f$  equals the turning index of  $\sigma^{n-1}$  under  $f'$ , and hence is not zero. But this contradicts Lemma 4. Thus, Theorem 1 is true.

**THEOREM 2.** *Let  $\eta^n \subset R^n$  be a closed  $n$ -cell with boundary  $\sigma^{n-1}$  and  $f$  a continuous map of  $\eta^n$  into  $R^n$  which leaves no point of  $\sigma^{n-1}$  fixed. If there exists an inner point  $e$  of  $\eta^n$  and an angle  $\alpha$  with  $0 \leq \alpha \leq \pi$ , such that for no point  $\sigma \in \sigma^{n-1}$  is  $\alpha$  an angle from the vector  $\overrightarrow{\sigma, f(\sigma)}$  to the vector  $\overrightarrow{e, \sigma}$  then  $f$  leaves at least one point fixed.*

*Proof.* Suppose  $f$  leaves no point fixed. We shall show that under the hypotheses of Theorem 2, either

i) for no point  $\sigma \in \sigma^{n-1}$  is the direction from  $\sigma$  to  $f(\sigma)$  opposite to that from  $e$  to  $\sigma$ ,

or

ii) for no point  $\sigma \in \sigma^{n-1}$  is the direction from  $\sigma$  to  $f(\sigma)$  opposite to that from  $\sigma$  to  $e$ .

For, otherwise we would have points  $\sigma_1$  and  $\sigma_2 \in \sigma^{n-1}$  such that, as  $\sigma$  traverses a path from  $\sigma_1$  to  $\sigma_2$  on  $\sigma^{n-1}$ , the angle between  $\overrightarrow{\sigma, f(\sigma)}$  and  $\overrightarrow{\sigma, e}$  would change continuously from 0 to  $\pi$ , hence assume the value  $\alpha$ , a contradiction.

If i) holds, we apply Lemma 1 taking the mapping  $g$  of Lemma 1 as the mapping  $f$ , and as the mapping  $h$ , we take a mapping which makes correspond to each point  $\sigma \in \sigma^{n-1}$  the intersection of the half line starting at the point  $e$  and passing through the point  $\sigma$ , with an  $(n-1)$ -sphere  $V^{n-1}$  whose center is  $e$  and which is located completely outside of  $\sigma^{n-1}$ . We infer by Lemma 1 that the turning indices of  $\sigma^{n-1}$  under  $f$  and  $h$  are equal. Since the turning index of  $\sigma^{n-1}$  under  $h$  clearly equals the turning index of  $\sigma^{n-1}$  relative to  $V^{n-1}$ , we infer from Lemma 3 that the turning index of  $\sigma^{n-1}$  under  $f$  is non-zero.

If ii) holds, again by Lemmas 1 and 3 the turning index of  $\sigma^{n-1}$  under  $f$  is non-zero. (Here, for the mapping  $g$  of Lemma 1, we again take the mapping  $f$ , and for the mapping  $h$ , we take a mapping which makes correspond to each point  $\sigma \in \sigma^{n-1}$  the intersection of the half line starting at the point  $e$  and passing through the point  $\sigma$ , with an  $(n-1)$ -sphere  $V^{n-1}$  whose center is  $e$  and which is located completely inside of  $\sigma^{n-1}$ ).

In short, the turning index of  $\sigma^{n-1}$  under the assumption of the absence of fixed points is non-zero, a fact which contradicts Lemma 4. Hence  $f$  has at least one fixed point, and Theorem 2 is proved.

**COROLLARY 1.** *Let  $E^n$  be a closed solid  $n$ -sphere and  $f$  a continuous mapping of  $E^n$  into  $R^n$  such that  $f$  maps the boundary  $S^{n-1}$  of  $E^n$  into  $E^n$ . Then  $f$  has at least one fixed point.*

*Proof.* If no point of  $S^{n-1}$  is fixed, then the hypotheses of Theorem 2 are seen to be satisfied with  $e$  at the center of the sphere  $E^n$  and  $\alpha = 0$ .

Clearly, Corollary 1 also follows immediately from Theorem 1. Proofs of this corollary also appear in the literature ([3], page 115).

**COROLLARY 2.** *Let  $\eta^n \subset R^n$  be a closed  $n$ -cell with boundary  $\sigma^{n-1}$ , and  $f$  and  $g$  two continuous maps of  $\eta^n$  into  $R^n$  such that for no point  $\sigma \in \sigma^{n-1}$  is  $f(\sigma) = g(\sigma)$ . If there exists an inner point  $e$  of  $\eta^n$  and a constant angle  $\beta$ ,  $0 \leq \beta \leq \pi$ , such that for no point  $\sigma \in \sigma^{n-1}$  is  $\beta$  an angle between the vectors  $e, \sigma$  and  $f(\sigma), g(\sigma)$ , then there is a point  $\eta_0 \in \eta^n$  such that  $f(\eta_0) = g(\eta_0)$ .*

*Proof.* Consider the map  $h$  of  $\eta^n$  into  $R^n$  such that for every point  $\eta \in \eta^n$  the vectors  $\eta, h(\eta)$  and  $f(\eta), g(\eta)$  are equal. By Theorem 2, the map  $h$  has a fixed point  $\eta_0$ . Consequently,  $f(\eta_0) = g(\eta_0)$ .

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