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$R^n$ , and  $(n-1)$ -spheres oriented with orientations induced by their interiors.

Symbols  $c^{n-1}$ ,  $g^{n-1}$ , ... denote oriented  $(n-1)$ -cycles in  $R^n$ ;  $D^{n-1}$ ,  $V^{n-1}$ , ... denote  $(n-1)$ -spheres in  $R^n$ .  $E^n$  denotes a closed solid  $n$ -sphere in  $R^n$ , and the boundary of  $E^n$  is denoted by  $S^{n-1}$ .  $\eta^n$  denotes a closed  $n$ -cell in  $R^n$  and the boundary of  $\eta^n$  is denoted by  $\sigma^{n-1}$ .

In this paper  $\eta^n$  is assumed to be the image of  $E^n$  under homeomorphism  $\theta$ , and  $\eta^n$  and  $\sigma^{n-1}$  obtain their orientations from  $E^n$  and  $S^{n-1}$  respectively.

### 3. THE TURNING INDEX

Let  $c^{n-1}$  be an  $(n-1)$ -cycle in  $R^n$  and  $g$  a continuous map of  $c^{n-1}$  into  $R^n$  having no fixed point. Let  $D^{n-1}$  be an  $(n-1)$ -sphere with center 0, called a *direction sphere* [2]. Let  $c^{n-1}$  be mapped on  $D^{n-1}$  as follows. To a point  $c \in c^{n-1}$  there corresponds a point  $d \in D^{n-1}$  such that the line segment from 0 to  $d$  has the same sense and direction as that from  $c$  to  $g(c)$ . The resulting  $(n-1)$ -cycle  $g^{n-1}$  on  $D^{n-1}$  is called, in the sequel, *the  $(n-1)$ -cycle  $g^{n-1}$  resulting from  $g$  applied to  $c^{n-1}$* , and the degree of the resulting map, that is, the multiple of  $D^{n-1}$  which is homologous to  $g^{n-1}$  (which is clearly independent of the radius of  $D^{n-1}$  and the location of 0) is called the *turning index* of  $c^{n-1}$  under  $g$ .

If  $p$  is a point not on  $c^{n-1}$ , the *index of  $p$  relative to  $c^{n-1}$*  is defined as the turning index of the map which maps every point of  $c^{n-1}$  into  $p$ . (For odd  $n$ , this is the negative of the corresponding definition given in [3], as shown by Theorem 1.5, page 105).

### 4. PRELIMINARY LEMMAS

LEMMA 1. *Let  $g$  and  $h$  be two continuous maps into  $R^n$  of an  $(n-1)$ -cycle  $c^{n-1}$ , such that neither leaves any point of  $c^{n-1}$  fixed, and, for no point  $c \in c^{n-1}$  are the directions from  $c$  to  $g(c)$  and from  $c$  to  $h(c)$  exactly opposite. Then the turning indices of  $c^{n-1}$  under  $g$  and  $h$  are equal.*

*Proof.* For each  $c \in c^{n-1}$ , the directions of the two vectors  $\overrightarrow{c,g(c)}$  and  $\overrightarrow{c,h(c)}$  are not opposite and hence, if not identical,

determine a 2-plane  $P$  in which they make an angle of less than  $\pi$  radians. As a parameter  $p$  varies from 0 to 1, let the direction of  $\overline{c, h(c)}$  change in  $P$  so that the angle between the two vectors  $\overline{c, h(c)}$  and  $\overline{c, g(c)}$  decreases uniformly to zero while their lengths remain fixed. If the angle is zero at the start, no change in direction takes place. For each value of  $p$ ,  $0 \leq p \leq 1$ , the corresponding mapping as determined above in the definition of turning index, maps  $c^{n-1}$  on the direction sphere  $D^{n-1}$ , and the result, as  $p$  varies from 0 to 1, is to deform the  $(n-1)$ -cycle  $h^{n-1}$  on  $D^{n-1}$  resulting from  $h$  applied to  $c^{n-1}$  into the  $(n-1)$ -cycle  $g^{n-1}$  resulting from  $g$  applied to  $c^{n-1}$ . Hence  $h^{n-1}$  is homologous to  $g^{n-1}$ , and therefore to the same multiple of  $D^{n-1}$ , so that the turning indices under consideration are equal. Thus Lemma 1 is proved.

LEMMA 2. *Let  $g$  be a continuous map into  $R^n$  of an  $(n-1)$ -cycle  $c^{n-1}$ , such that  $c^{n-1}$  and  $g(c^{n-1})$  are contained in different half-spaces into which  $R^n$  is separated by some  $(n-1)$ -plane. Then the turning index of  $c^{n-1}$  under  $g$  is zero.*

*Proof.* Since the  $(n-1)$ -cycle  $g^{n-1}$  resulting from  $g$  applied to  $c^{n-1}$  is clearly entirely on one hemisphere of  $D^{n-1}$ , we conclude that  $c^{n-1}$  cannot be homologous to any multiple of  $D^{n-1}$  other than zero. Thus Lemma 2 is proved.

LEMMA 3. *Let  $\sigma^{n-1}$  be the boundary of a closed  $n$ -cell  $\eta^n \subset R^n$ . Let  $e$  be a point in the inside of  $\sigma^{n-1}$ . Then the index of  $e$  relative to  $\sigma^{n-1}$  is 1 or  $-1$ .*

While this result is given in [3], page 109, Theorem 4.1, the following proof is given as shorter and obtained independently.

*Proof.* Let  $\eta^n$  and  $\sigma^{n-1}$  be respectively the homeomorphic images (under homeomorphism  $\theta$ ) of the closed solid  $n$ -sphere  $E^n$  with boundary  $S^{n-1}$ , i.e.,  $\eta^n = \theta(E^n)$  and  $\sigma^{n-1} = \theta(S^{n-1})$ . By use of the invariance of regionality, it is easy to show that  $\eta^n = \theta(E^n)$  contains no point outside  $\sigma^{n-1}$  and contains every point inside  $\sigma^{n-1}$ .

Let  $V^{n-1}$  be an  $(n-1)$ -sphere with center at  $e$ , so small that  $V^{n-1}$  and its interior are inside  $\sigma^{n-1}$ , hence composed of points of  $\eta^n$ . Let  $\beta^{n-1} = \theta^{-1}(V^{n-1})$  and  $d = \theta^{-1}(e)$ .

For each point  $b \in \beta^{n-1}$  let the half-line beginning at  $d$  and passing through  $b$  intersect  $S^{n-1}$  at  $b'$ .

Now, for every  $t$ , with  $0 \leq t \leq 1$ , let  $\beta^{n-1}(t)$  be the  $(n-1)$ -cycle determined as follows. For each point  $b \in \beta^{n-1}$  there corresponds a point  $b(t)$  of  $\beta^{n-1}(t)$  on the closed segment from  $b$  to  $b'$  such that the distance from  $b$  to  $b(t)$  is  $t$  times the distance from  $b$  to  $b'$ .

Let  $V^{n-1}(t) = \theta[\beta^{n-1}(t)]$ ,  $0 \leq t \leq 1$ .

As  $t$  varies from 0 to 1, the cycle  $V^{n-1}(t)$  undergoes a deformation from initial position  $V^{n-1}(0) = V^{n-1}$  to final position  $V^{n-1}(1)$ . Since  $V^{n-1}(1)$  is on  $\sigma^{n-1}$ , there is an integer  $x$  such that

$$(1) \quad V^{n-1}(1) \sim x \sigma^{n-1} \quad \text{on } \sigma^{n-1},$$

where  $\sim$  stands for "is homologous to".

For each  $t$ , let  $k(t)$  be the mapping which maps every point of  $V^{n-1}(t)$  into  $e$ , and let  $V^{n-1}$  serve as the direction sphere. As  $t$  varies from 0 to 1, the  $(n-1)$ -cycle  $k^{n-1}(0)$  resulting from  $k(0)$  applied to  $V^{n-1}$  is deformed on the direction sphere  $V^{n-1}$  into the  $(n-1)$ -cycle  $k^{n-1}(1)$  resulting from  $k(1)$  applied to  $V^{n-1}(1)$ . Thus these two  $(n-1)$  cycles are homologous on  $V^{n-1}$ . Therefore the index of  $e$  relative to  $V^{n-1}$  equals the index of  $e$  relative to  $V^{n-1}(1)$ . However, since  $k(0)$  maps every point of  $V^{n-1}$  into  $e$ , we derive that ([4], page 92)

$$(2) \quad \text{the index of } e \text{ relative to } V^{n-1}(1) = (-1)^n.$$

Let  $y$  be the index of  $e$  relative to  $\sigma^{n-1}$ . From (1) we infer that  $xy$  is the index of  $e$  relative to  $V^{n-1}(1)$ . Hence, by (2),  $xy = (-1)^n$ . Consequently,  $y = 1$  or  $y = -1$ . Thus Lemma 3 is proved.

LEMMA 4. *If a continuous map  $f$  of a closed  $n$ -cell  $\eta^n \subset R^n$  into  $R^n$  has no fixed point, then the turning index of the boundary  $\sigma^{n-1}$  of  $\eta^n$  under  $f$  is zero.*

*Proof.* Let, as in the proof of Lemma 3,  $\eta^n = \theta(E^n)$  and  $\sigma^{n-1} = \theta(S^{n-1})$  be respectively the images under the homeomorphism  $\theta$  of the closed solid  $n$ -sphere  $E^n$  and its boundary  $S^{n-1}$ .

Let  $u$  be the center of  $S^{n-1}$ . Since  $f$  has no fixed point, it is clear that we can choose  $d > 0$  so small that a closed solid  $n$ -sphere  $H_d^n$  of radius  $d$  with center at  $\theta(u)$  is entirely in  $\eta^n$ , and  $H_d^n$  and its image  $f(H_d^n)$  are contained in different half-spaces into which  $R^n$  is separated by some  $(n-1)$ -plane.

Now, let  $S^{n-1}$  undergo a deformation by uniform radial shrinking toward  $u$  till it reaches a position  $S_2^{n-1}$  whose image  $\sigma_2^{n-1}$  under  $\theta$  is contained in the interior of  $H_d^n$ . By means of  $\theta$ , there results a deformation of  $\sigma^{n-1}$  into  $\sigma_2^{n-1}$  which by means of the mapping  $f$  induces a deformation, on the direction sphere, of the  $(n-1)$ -cycle  $f^{n-1}$  resulting from  $f$  applied to  $\sigma^{n-1}$  into the  $(n-1)$ -cycle  $f_2^{n-1}$  resulting from  $f$  applied to  $\sigma_2^{n-1}$ .

Thus the turning index of  $\sigma^{n-1}$  under  $f$  equals the turning index of  $\sigma_2^{n-1}$  under  $f$ , which by Lemma 2 equals zero. Thus Lemma 4 is proved.

## 5. THE THEOREMS

**THEOREM 1.** *Let  $\eta^n \subset R^n$  be a closed  $n$ -cell and  $f$  a continuous mapping of  $\eta^n$  into  $R^n$  such that  $f$  maps the boundary  $\sigma^{n-1}$  of  $\eta^n$  into  $\eta^n$ . Then  $f$  has at least one fixed point.*

*Proof.* Assume no fixed points. Let, as in the case of Lemma 3,  $\eta^n$  and  $\sigma^{n-1}$  be respectively the images (under the homeomorphism  $\theta$ ) of the closed solid  $n$ -sphere  $E^n$  with boundary  $S^{n-1}$ , i.e.,  $\eta^n = \theta(E^n)$  and  $\sigma^{n-1} = \theta(S^{n-1})$ .

Let  $u$  be the center of  $S^{n-1}$ . Consider the mapping  $f'$  of  $\sigma^{n-1}$  which maps every point  $\sigma \in \sigma^{n-1}$  into the point  $\theta(u)$ . Since  $f'$  is the mapping which appears in the definition of the index of  $\theta(u)$  relative to  $\sigma^{n-1}$ , we see by Lemma 3 that the turning index of  $\sigma^{n-1}$  under  $f'$  is non-zero.

By hypothesis,  $f(\sigma) \in \eta^n$  for every  $\sigma \in \sigma^{n-1}$ . Hence we may deform  $f(\sigma^{n-1})$  as follows. As a parameter  $p$  varies from 0 to 1, the point  $\sigma'$  moves in  $\eta^n$  along the path  $\overline{\theta[\theta^{-1}f(\sigma), u]}$  starting from  $\sigma$  and ending at  $\theta(u)$ .

For  $p = 1$ , the above deformation yields the mapping  $f'$ . Therefore, the  $(n-1)$ -cycle resulting from  $f$  applied to  $\sigma^{n-1}$  is homologous on the direction sphere (as a consequence of a deformation) to the  $(n-1)$ -cycle resulting from  $f'$  applied to