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This result is due to Pontrjagin, Thom, Milnor, Averbuh, and Wall. (See [2, 9, 19].) For the definition of the Pontrjagin numbers $p_{i_1} \dots p_{i_n}[V] \in J$ the reader is refered to Hirzebruch [6]. These numbers are defined only if the dimension k is a multiple of 4.

The *oriented cobordism ring* $\Omega_* = H_*(\mathscr{D}_o)$ is defined as follows. For $V \in \mathscr{D}_o$ let $-V$ denote the same manifold V with the opposite orientation. We will say that

$$V \equiv V' \quad (mod\ \partial\mathscr{D}_o)$$

if $(-V) + V'$ is the boundary of some manifold in $\mathscr{D}_o$. As an example, for any closed manifold V we have $V \equiv V \pmod{\partial\mathscr{D}_o}$ since

$$(-V) + V \approx \partial(V \times I)$$

where I denotes the unit interval. The set of all such congruence classes form the required group Ω_k. Again the cartesian product operation makes $\Omega_* = (\Omega_0, \Omega_1, \dots)$ into a graded ring.

It follows from Theorem 1′ that Ω_k is a finitely generated group of the form

$$J \oplus \dots \oplus J \oplus J_2 \oplus \dots \oplus J_2$$

where infinite cyclic summands can occur only if $k \equiv 0 \pmod 4$.

THEOREM 2′. — *The ring* Ω_*, *modulo the ideal consisting of* 2-*torsion elements, is a polynomial ring* $J[Y_4, Y_8, Y_{12}, \dots]$ *with one generator in each dimension divisible by* 4.

The complex projective space of real dimension $4m$ can be taken as generator for $m = 1, 2, 3$. However a different generator is needed in dimension 16.

For a description of the 2-torsion in Ω_* the reader is referred to Wall's paper.

2. MANIFOLDS WITH X-STRUCTURE.

In this section we will define the concept of an " X-structure " on the tangent bundle of a differentiable manifold; and study the corresponding cobordism theory.

First recall Steenrod's definition of a tensor field [15, § 6.4 and § 9.1 with mild alterations]. Every differentiable k-manifold V can be made Riemannian and hence has a tangent bundle with structural group O_k. Let X be any topological space on which the group O_k acts. Then we can form the weakly associated bundle with base space V and fibre X. This may be called the " tensor bundle of type X " and its cross-sections are " tensor fields ". As an example, if $k = 2m$, then O_{2m} acts on the coset space O_{2m}/U_m.

A cross-section of the corresponding bundle is called a *quasi-* (or almost) *complex structure* on V. (See [15, § 41.10].)

We will modify this definition as follows, so that it makes sense for all dimensions simultaneously. Let O denote the union of the orthogonal groups $O_1 \subset O_2 \subset O_3 \subset \ldots$ in the fine topology. Then we require that this infinite orthogonal group O act on the space X. It follows that each O_k acts on X. Hence there is a tensor bundle of type X over any manifold $V \in \mathfrak{D}$.

Definition: A homotopy class of cross-sections of the tensor bundle with fibre X over V is called an *X-structure* on V. A manifold $V \in \mathfrak{D}$ together with an X-structure on V is called an *X-manifold.* We will still use the single symbol V to denote this pair.

Now if V is an X-manifold then ∂V is also. Given any closed X-manifold V one can define a second X-manifold $-V$ so that

$$\partial(V \times I) \approx V + (-V) .$$

Thus one can define a cobordism group for the class of X-manifolds. The resulting group will be denoted by $N_k(X)$ and called the X-*cobordism group.* (Following Atiyah [1] this could also be called the k-th " bordism group " of the O-space X.)

Example 1. Let O/U denote the union of the spaces

$$O_2/U_1 \subset O_4/U_2 \subset O_6/U_3 \subset \ldots$$

in the fine topology with O acting on O/U in the usual way. Then a manifold with an O/U-structure will be called a *weakly complex manifold.* (Compare Hirzebruch [7].) For example any complex manifold is quasi-complex and hence weakly complex. Any sphere can be given an O/U-structure although only S^2 and S^6 possess quasi-complex structures.

The following results are due to Milnor and Novikov.

THEOREM 1''. — *A closed weakly complex manifold* V *is the boundary of a weakly complex manifold if and only if its Chern numbers* $c_{i_1} \dots c_{i_n}$ [V] *are all zero.*

(Explanation: an O/U-structure on V determines a preferred U-bundle over V. Hence Chern classes are defined.) It follows that $N_k(O/U)$ is zero for k odd and is free abelian for k even.

THEOREM 2''. — *The graded group* $N_*(O/U)$ *has a natural ring structure, making it into a polynomial ring* $J[Y_2, Y_4, Y_6, \dots]$ *with one generator in each even dimension.*

As generators one can take certain algebraic varieties with their natural complex structures. (Compare [7]. It is not known whether connected varieties will suffice.)

Example 2. More generally one could use any subgroup G of the infinite orthogonal group in place of U. For example using the infinite symplectic group Sp we would obtain a cobordism ring $N_*(O/Sp)$ which is appropriate for the study of " weakly quaternionic manifolds ". The following six groups seem particularly interesting:

$$1 \subset Sp \subset SU \subset U \subset SO \subset 0 .$$

Starting from the right, the ring $N_*(O/O)$ is just the non-oriented cobordism ring N_* and $N_*(O/SO)$ is the oriented cobordism ring Ω_*. The rings $N_*(O/SU)$ and $N_*(O/Sp)$ are more or less unknown. (Compare the concluding remarks in [9].)

The ring $N_*(O/1) = N_*(O)$ has essentially been studied by Pontrjagin [11]. An O-structure on V is a trivialization of the tangent O-bundle of V (the " stable " tangent bundle). Manifolds which admit such a structure are called " π-manifolds ". It turns out that $N_k(O)$ is isomorphic to the stable homotopy groups $\pi_{k+n}(S^n)$ of the n-sphere, with n large. This fact is the basis for Pontrjagin's method of studying homotopy groups.

Example 3. Let X be a space on which O operates trivially. Then an X-structure on V is just a prefered homotopy class of maps $V \to X$. As cases of particular interest X might be an Eilenberg-MacLane space or the classifying space of a group. How does one compute he groups $N_k(X)$?

The above definitions can be modified slightly by admitting only oriented manifolds. Thus one obtains groups $\Omega_k(X)$ where X is any space on which the rotation group SO acts. Again I do not know how to compute these groups. (Added in proof: See Conner and Floyd [21].)

Example 4. Let P denote the infinite real projective space, with the infinite rotation group SO acting in the natural way. The cobordism groups $\Omega_k(P)$ for oriented manifolds with P-structure can be called the *spinor cobordism groups.* This name is appropriate since a P-structure is roughly a " lifting " of the structural group of the tangent bundle to the infinite spinor group. A manifold admits a P-structure if and only if its Stiefel-Whitney class w_2 is zero. The groups $\Omega_k(P)$ have no odd torsion, but otherwise I do not know much about them.

3. Miscellaneous cobordism theories.

So far we have concentrated on differentiable manifolds. However one could equally well define a cobordism group based on the class $\mathscr{T}$ of all compact topological manifolds. (Compare Brown [3, Theorem 3].) The natural correspondence $\mathscr{D} \to \mathscr{T}$ induces a homomorphism from the differentiable cobordism group $N_k = H_k(\mathscr{D})$ to the topological cobordism group $H_k(\mathscr{T})$.

Since Thom [16] has shown that Stiefel-Whitney classes can be defined topologically, we have:

Theorem 3 (Thom). — *The homomorphism* $\mathrm{N}_k \to H_k(\mathscr{T})$ *has kernel zero.*

Problem: Is this homomorphism onto ?

Another possibility would be to consider the class $\mathscr{C}_0$ of all compact, oriented, combinatorial manifolds. Whitehead [20] has shown that each differentiable manifold has a preferred class of triangulations. Hence there is a natural homomorphism from