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This result is due to Pontrjagin, Thom, Milnor, Averbuh, and Wall. (See [2, 9, 19].) For the definition of the Pontrjagin numbers  $p_{i_1} \dots p_{i_n} [V] \in J$  the reader is referred to Hirzebruch [6]. These numbers are defined only if the dimension  $k$  is a multiple of 4.

The *oriented cobordism ring*  $\Omega_* = H_*(\mathcal{D}_o)$  is defined as follows. For  $V \in \mathcal{D}_o$  let  $-V$  denote the same manifold  $V$  with the opposite orientation. We will say that

$$V \equiv V' \pmod{\partial \mathcal{D}_o}$$

if  $(-V) + V'$  is the boundary of some manifold in  $\mathcal{D}_o$ . As an example, for any closed manifold  $V$  we have  $V \equiv V \pmod{\partial \mathcal{D}_o}$  since

$$(-V) + V \approx \partial(V \times I)$$

where  $I$  denotes the unit interval. The set of all such congruence classes form the required group  $\Omega_k$ . Again the cartesian product operation makes  $\Omega_* = (\Omega_0, \Omega_1, \dots)$  into a graded ring.

It follows from Theorem 1' that  $\Omega_k$  is a finitely generated group of the form

$$J \oplus \dots \oplus J \oplus J_2 \oplus \dots \oplus J_2$$

where infinite cyclic summands can occur only if  $k \equiv 0 \pmod{4}$ .

**THEOREM 2'.** — *The ring  $\Omega_*$ , modulo the ideal consisting of 2-torsion elements, is a polynomial ring  $J[Y_4, Y_8, Y_{12}, \dots]$  with one generator in each dimension divisible by 4.*

The complex projective space of real dimension  $4m$  can be taken as generator for  $m = 1, 2, 3$ . However a different generator is needed in dimension 16.

For a description of the 2-torsion in  $\Omega_*$  the reader is referred to Wall's paper.

## 2. MANIFOLDS WITH $X$ -STRUCTURE.

In this section we will define the concept of an " $X$ -structure" on the tangent bundle of a differentiable manifold; and study the corresponding cobordism theory.

First recall Steenrod's definition of a tensor field [15, § 6.4 and § 9.1 with mild alterations]. Every differentiable  $k$ -manifold  $V$  can be made Riemannian and hence has a tangent bundle with structural group  $O_k$ . Let  $X$  be any topological space on which the group  $O_k$  acts. Then we can form the weakly associated bundle with base space  $V$  and fibre  $X$ . This may be called the "tensor bundle of type  $X$ " and its cross-sections are "tensor fields". As an example, if  $k = 2m$ , then  $O_{2m}$  acts on the coset space  $O_{2m}/U_m$ .

A cross-section of the corresponding bundle is called a *quasi*-(or almost) *complex structure* on  $V$ . (See [15, § 41.10].)

We will modify this definition as follows, so that it makes sense for all dimensions simultaneously. Let  $O$  denote the union of the orthogonal groups  $O_1 \subset O_2 \subset O_3 \subset \dots$  in the fine topology. Then we require that this infinite orthogonal group  $O$  act on the space  $X$ . It follows that each  $O_k$  acts on  $X$ . Hence there is a tensor bundle of type  $X$  over any manifold  $V \in \mathcal{D}$ .

*Definition:* A homotopy class of cross-sections of the tensor bundle with fibre  $X$  over  $V$  is called an  $X$ -structure on  $V$ . A manifold  $V \in \mathcal{D}$  together with an  $X$ -structure on  $V$  is called an  $X$ -manifold. We will still use the single symbol  $V$  to denote this pair.

Now if  $V$  is an  $X$ -manifold then  $\partial V$  is also. Given any closed  $X$ -manifold  $V$  one can define a second  $X$ -manifold  $-V$  so that

$$\partial(V \times I) \approx V + (-V).$$

Thus one can define a cobordism group for the class of  $X$ -manifolds. The resulting group will be denoted by  $N_k(X)$  and called the  $X$ -cobordism group. (Following Atiyah [1] this could also be called the  $k$ -th "bordism group" of the  $O$ -space  $X$ .)

*Example 1.* Let  $O/U$  denote the union of the spaces

$$O_2/U_1 \subset O_4/U_2 \subset O_6/U_3 \subset \dots$$

in the fine topology with  $O$  acting on  $O/U$  in the usual way. Then a manifold with an  $O/U$ -structure will be called a *weakly complex manifold*. (Compare Hirzebruch [7].) For example

any complex manifold is quasi-complex and hence weakly complex. Any sphere can be given an  $O/U$ -structure although only  $S^2$  and  $S^6$  possess quasi-complex structures.

The following results are due to Milnor and Novikov.

**THEOREM 1''.** — *A closed weakly complex manifold  $V$  is the boundary of a weakly complex manifold if and only if its Chern numbers  $c_{i_1} \dots c_{i_n}[V]$  are all zero.*

(Explanation: an  $O/U$ -structure on  $V$  determines a preferred  $U$ -bundle over  $V$ . Hence Chern classes are defined.) It follows that  $N_k(O/U)$  is zero for  $k$  odd and is free abelian for  $k$  even.

**THEOREM 2''.** — *The graded group  $N_*(O/U)$  has a natural ring structure, making it into a polynomial ring  $J[Y_2, Y_4, Y_6, \dots]$  with one generator in each even dimension.*

As generators one can take certain algebraic varieties with their natural complex structures. (Compare [7]. It is not known whether connected varieties will suffice.)

*Example 2.* More generally one could use any subgroup  $G$  of the infinite orthogonal group in place of  $U$ . For example using the infinite symplectic group  $Sp$  we would obtain a cobordism ring  $N_*(O/Sp)$  which is appropriate for the study of "weakly quaternionic manifolds". The following six groups seem particularly interesting:

$$1 \subset Sp \subset SU \subset U \subset SO \subset 0.$$

Starting from the right, the ring  $N_*(O/O)$  is just the non-oriented cobordism ring  $N_*$  and  $N_*(O/SO)$  is the oriented cobordism ring  $\Omega_*$ . The rings  $N_*(O/SU)$  and  $N_*(O/Sp)$  are more or less unknown. (Compare the concluding remarks in [9].)

The ring  $N_*(O/1) = N_*(O)$  has essentially been studied by Pontrjagin [11]. An  $O$ -structure on  $V$  is a trivialization of the tangent  $O$ -bundle of  $V$  (the "stable" tangent bundle). Manifolds which admit such a structure are called " $\pi$ -manifolds". It turns out that  $N_k(O)$  is isomorphic to the stable homotopy groups  $\pi_{k+n}(S^n)$  of the  $n$ -sphere, with  $n$  large. This fact is the basis for Pontrjagin's method of studying homotopy groups.

*Example 3.* Let  $X$  be a space on which  $O$  operates trivially. Then an  $X$ -structure on  $V$  is just a preferred homotopy class of maps  $V \rightarrow X$ . As cases of particular interest  $X$  might be an Eilenberg-MacLane space or the classifying space of a group. How does one compute the groups  $N_k(X)$ ?

The above definitions can be modified slightly by admitting only oriented manifolds. Thus one obtains groups  $\Omega_k(X)$  where  $X$  is any space on which the rotation group  $SO$  acts. Again I do not know how to compute these groups. (Added in proof: See Conner and Floyd [21].)

*Example 4.* Let  $P$  denote the infinite real projective space, with the infinite rotation group  $SO$  acting in the natural way. The cobordism groups  $\Omega_k(P)$  for oriented manifolds with  $P$ -structure can be called the *spinor cobordism groups*. This name is appropriate since a  $P$ -structure is roughly a "lifting" of the structural group of the tangent bundle to the infinite spinor group. A manifold admits a  $P$ -structure if and only if its Stiefel-Whitney class  $w_2$  is zero. The groups  $\Omega_k(P)$  have no odd torsion, but otherwise I do not know much about them.

### 3. MISCELLANEOUS COBORDISM THEORIES.

So far we have concentrated on differentiable manifolds. However one could equally well define a cobordism group based on the class  $\mathcal{T}$  of all compact topological manifolds. (Compare Brown [3, Theorem 3].) The natural correspondence  $\mathcal{D} \rightarrow \mathcal{T}$  induces a homomorphism from the differentiable cobordism group  $N_k = H_k(\mathcal{D})$  to the topological cobordism group  $H_k(\mathcal{T})$ .

Since Thom [16] has shown that Stiefel-Whitney classes can be defined topologically, we have:

**THEOREM 3 (Thom).** — *The homomorphism  $N_k \rightarrow H_k(\mathcal{T})$  has kernel zero.*

**Problem:** Is this homomorphism onto?

Another possibility would be to consider the class  $\mathcal{C}_o$  of all compact, oriented, combinatorial manifolds. Whitehead [20] has shown that each differentiable manifold has a preferred class of triangulations. Hence there is a natural homomorphism from