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# A SURVEY OF COBORDISM THEORY<sup>1</sup>

by J. MILNOR

This paper will start out with a discussion of known results and then will taper off into a discussion of unsolved problems.

The theory of cobordism was initiated by L. Pontrjagin and V. A. Rohlin [10, 12]. It came of age with the work of R. Thom [17]. The basic question in this theory is the following. Let  $\mathcal{M}$  be some class of compact manifolds. Given  $V \in \mathcal{M}$  how can one decide whether or not  $V$  is the boundary of some other manifold in  $\mathcal{M}$ ? Of course a necessary condition is that  $V$  itself must be a *closed* manifold: that is the boundary  $\partial V$  must be *vacuous*.

## 1. THE CLASSICAL COBORDISM GROUPS $N_k$ AND $\Omega_k$ .

As a first illustration of this problem let  $\mathcal{D}$  denote the class of all compact differentiable manifolds. The manifolds  $V \in \mathcal{D}$  need not be connected or orientable, and are allowed to have boundaries.

**THEOREM 1** (Pontrjagin, Thom). — *A closed  $k$ -dimensional manifold  $V \in \mathcal{D}$  is the boundary of some  $(k + 1)$ -dimensional manifold in  $\mathcal{D}$  if and only if the Stiefel-Whitney numbers  $\omega_{i_1} \dots \omega_{i_n} [V]$  are all zero.*

(Explanation: The Stiefel-Whitney cohomology classes<sup>2)</sup>  $\omega_i \in H^i(V; J_2)$  are defined for example in Steenrød [15]. If  $i_1 + \dots + i_n = k$  is any partition of  $k$  then the cup product  $\omega_{i_1} \dots \omega_{i_n}$  is a top dimensional cohomology class. Applying the canonical "integration" homomorphism

$$[V]: H^k(V; J_2) \rightarrow J_2$$

we obtain a "Stiefel-Whitney number"  $\omega_{i_1} \dots \omega_{i_n} [V] \in J_2$ .)

<sup>1)</sup> Talk delivered at the Zurich Colloquium on Differential Geometry and Topology, June 1960.

<sup>2)</sup> The notation  $J$  will be used for the integers and  $J_2$  for the integers modulo 2.

The *non-oriented cobordism group*  $N_k = H_k(\mathcal{D})$  is constructed as follows. Given two  $k$ -manifolds  $V, V' \in \mathcal{D}$  the *sum*  $V + V'$  will mean the (disjoint) topological sum, provided with a differentiable structure in the obvious way.

*Definition.* Two closed manifolds  $V, V' \in \mathcal{D}$  are *congruent modulo*  $\mathcal{D}$  if  $V + V'$  is the boundary of some manifold in  $\mathcal{D}$ . The set of all congruence classes of closed  $k$ -manifolds, under the composition operation  $+$ , forms the required group  $N_k$ . We will also use the notation  $H_k(\mathcal{D})$  for this group since it is something like a homology group. (The Russian term for “cobordism” is “intrinsic homology”.)

It follows from Theorem 1 that each  $N_k$  is a finite abelian group of the form  $J_2 \oplus \dots \oplus J_2$ .

The cartesian product operation between differentiable manifolds gives rise to a bilinear pairing

$$N_k \oplus N_l \rightarrow N_{k+l}.$$

Thus the graded group  $N_* = (N_0, N_1, \dots)$  has the structure of a graded ring.

**THEOREM 2 (Thom).** — *The non-oriented cobordism ring  $N_*$  has the structure of a polynomial algebra*

$$J_2 [X_2, X_4, X_5, X_6, X_8, X_9, \dots]$$

*with one generator  $X_k \in N_k$  for each dimension which is not of the form  $2^m - 1$ .*

If  $k$  is even then the real projective  $k$ -space can be taken as generator. For  $k$  odd generators have been constructed by Dold [4].

Thom's proof of Theorems 1 and 2 involves a brilliant mixture of algebra and geometry. A key step in the argument is his proof that  $N_k$  is isomorphic to a certain homotopy group. I will not try to give details.

Next consider the class  $\mathcal{D}_o$  consisting of all *oriented* compact differentiable manifolds.

**THEOREM 1'.** — *A closed manifold in  $\mathcal{D}_o$  is the boundary of a manifold in  $\mathcal{D}_o$  if and only if both its Stiefel-Whitney numbers and its Pontrjagin numbers are zero.*

This result is due to Pontrjagin, Thom, Milnor, Averbuh, and Wall. (See [2, 9, 19].) For the definition of the Pontrjagin numbers  $p_{i_1} \dots p_{i_n} [V] \in J$  the reader is referred to Hirzebruch [6]. These numbers are defined only if the dimension  $k$  is a multiple of 4.

The *oriented cobordism ring*  $\Omega_* = H_*(\mathcal{D}_o)$  is defined as follows. For  $V \in \mathcal{D}_o$  let  $-V$  denote the same manifold  $V$  with the opposite orientation. We will say that

$$V \equiv V' \pmod{\partial \mathcal{D}_o}$$

if  $(-V) + V'$  is the boundary of some manifold in  $\mathcal{D}_o$ . As an example, for any closed manifold  $V$  we have  $V \equiv V \pmod{\partial \mathcal{D}_o}$  since

$$(-V) + V \approx \partial(V \times I)$$

where  $I$  denotes the unit interval. The set of all such congruence classes form the required group  $\Omega_k$ . Again the cartesian product operation makes  $\Omega_* = (\Omega_0, \Omega_1, \dots)$  into a graded ring.

It follows from Theorem 1' that  $\Omega_k$  is a finitely generated group of the form

$$J \oplus \dots \oplus J \oplus J_2 \oplus \dots \oplus J_2$$

where infinite cyclic summands can occur only if  $k \equiv 0 \pmod{4}$ .

**THEOREM 2'.** — *The ring  $\Omega_*$ , modulo the ideal consisting of 2-torsion elements, is a polynomial ring  $J[Y_4, Y_8, Y_{12}, \dots]$  with one generator in each dimension divisible by 4.*

The complex projective space of real dimension  $4m$  can be taken as generator for  $m = 1, 2, 3$ . However a different generator is needed in dimension 16.

For a description of the 2-torsion in  $\Omega_*$  the reader is referred to Wall's paper.

## 2. MANIFOLDS WITH X-STRUCTURE.

In this section we will define the concept of an "X-structure" on the tangent bundle of a differentiable manifold; and study the corresponding cobordism theory.