

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 7 (1961)  
**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE COHOMOLOGY ALGEBRA OF A SPACE  
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**Kapitel:** 9. Universal A-algebras  
**DOI:** <https://doi.org/10.5169/seals-37129>

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A trivial example is provided by any algebra  $X$  over  $R$ . Note first that  $\varphi: R \otimes R \rightarrow R$  defined by  $\varphi(r_1 \otimes r_2) = r_1 r_2$  is an isomorphism (recall that  $\otimes = \otimes_R$ ). Set  $\Psi = \varphi^{-1}: R \rightarrow R \otimes R$ , then  $\varphi, \Psi$  give a natural structure of a Hopf algebra to the ground ring  $R$ . It is easily checked that the natural  $R$ -structure in  $X \otimes X$  coincides with that defined by  $\Psi$ . Thus any algebra over the ground ring is an algebra over the ground ring regarded as a Hopf algebra.

As another example, let  $X$  be an algebra over  $R$ , and let  $\pi$  be a group of automorphisms of the algebra  $X$ . Let  $A$  be the group ring of  $\pi$  over  $R$  with the usual multiplication. Define the diagonal  $\Psi: A \rightarrow A \otimes A$  to be the mapping induced by the diagonal mapping  $d: \pi \rightarrow \pi \times \pi$ . Then  $A$  becomes a Hopf algebra. Since any  $g \in \pi$  is an automorphism,  $g(x_1 x_2) = (gx_1)(gx_2)$ ; and since  $dg = (g, g)$ , it follows that 8.1 holds. Thus any algebra is an algebra over the Hopf algebra of its automorphism group.

## 9. UNIVERSAL $A$ -ALGEBRAS.

The foregoing examples of algebras over Hopf algebras arose naturally. We now show how to construct them in a wholesale fashion.

Let  $A$  be any Hopf algebra. It is easy to construct many modules over the algebra  $A$  (i.e. take quotients of  $A$  by left ideals, and then take direct sums of these). Let  $M$  be any graded  $A$ -module. Let  $M^n$  denote the tensor product of  $n$  copies of  $M$ . As in section 7,  $M^n$  is an  $A$ -module. Form the direct sum

$$T(M) = \sum_{n=0}^{\infty} M^n$$

where  $M^0 = R$ . Define  $\mu: T(M) \otimes T(M) \rightarrow T(M)$  in terms of components  $x \in M^r, y \in M^s$  by  $\mu(x \otimes y) = x \otimes y \in M^{r+s}$  making use of the associative law  $M^r \otimes M^s \approx M^{r+s}$ . In this way  $T(M)$  is an associative algebra. It is called the *free associative algebra* generated by  $M$  (also, the *tensor algebra* of  $M$ ). Since the associative law  $M^r \otimes M^s \approx M^{r+s}$  is an  $A$ -mapping, it follows that  $T(M)$  is an algebra over the Hopf algebra  $A$ .

Form now the quotient of  $T(M)$  by the ideal  $N$  generated by elements

$$(9.2) \quad x \otimes y - (-1)^{pq} y \otimes x \text{ where } x \in M_p, \quad y \in M_q.$$

The quotient, denoted by  $U(M)$ , is called the *free, commutative and associative algebra generated by M*. If we assume that the diagonal mapping  $\Psi$  of  $A$  is commutative, then it is readily verified that  $N$  is an  $A$ -submodule of  $T(M)$ . Hence  $U(M)$  becomes an algebra over the Hopf algebra  $A$ .

As is well known, the algebra  $T(M)$  is *universal* in the sense that any  $R$ -mapping of  $M$  into an algebra  $X$  extends to a unique mapping of algebras  $T(M) \rightarrow X$ . Furthermore, if  $X$  is an algebra over  $A$ , and  $M \rightarrow X$  is an  $A$ -mapping, so also is  $T(M) \rightarrow X$ . A similar statement holds for  $U(M)$  in case  $X$  is commutative.

Additional algebras over  $A$  can be constructed by taking a submodule of  $T(M)$  or  $U(M)$  forming the  $A$ -ideal it generates, and passing to the quotient algebra. It is easily seen that any  $A$ -algebra can be obtained as such a quotient.

In the special case where  $A$  is the algebra  $\mathcal{A}_p$  of reduced powers, only certain  $M$ 's are admissible, namely, those which satisfy the dimensionality restriction 4.9:  $\mathcal{P}^i x = 0$  whenever  $2i > \dim x$ . Moreover, in forming  $U(M)$ , we must increase the ideal  $N$  so as to include all elements of the form

$$(9.3) \quad \mathcal{P}^k x - (x \otimes x \otimes \dots \otimes x) \text{ (} p \text{ factors)}, \quad x \in M_{2k}.$$

This insures that the relation 4.8, namely,  $\mathcal{P}^k y = y^p$  is valid for  $y \in U(M)_{2k}$ . (It is a pleasant exercise in the use of the Adem-Cartan relations to show that  $N$  is an  $\mathcal{A}_p$ -module.) With these modifications, the resulting  $U(M)$  is meaningful for algebraic topology.

## 10. REFORMULATION OF THE PROBLEM.

We are now in a position to formulate a problem similar to the one posed in section 2, but having a better chance of a positive solution. Recall that the algebra  $F(R, q)^\infty$  of section 2 is small in that it has a single generator but is otherwise as big as