

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 7 (1961)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE COHOMOLOGY ALGEBRA OF A SPACE
Autor: Steenrod, N. E.
Kapitel: 9. Universal A-algebras
DOI: <https://doi.org/10.5169/seals-37129>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 09.07.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

A trivial example is provided by any algebra X over R . Note first that $\varphi: R \otimes R \rightarrow R$ defined by $\varphi(r_1 \otimes r_2) = r_1 r_2$ is an isomorphism (recall that $\otimes = \otimes_R$). Set $\Psi = \varphi^{-1}: R \rightarrow R \otimes R$, then φ, Ψ give a natural structure of a Hopf algebra to the ground ring R . It is easily checked that the natural R -structure in $X \otimes X$ coincides with that defined by Ψ . Thus any algebra over the ground ring is an algebra over the ground ring regarded as a Hopf algebra.

As another example, let X be an algebra over R , and let π be a group of automorphisms of the algebra X . Let A be the group ring of π over R with the usual multiplication. Define the diagonal $\Psi: A \rightarrow A \otimes A$ to be the mapping induced by the diagonal mapping $d: \pi \rightarrow \pi \times \pi$. Then A becomes a Hopf algebra. Since any $g \in \pi$ is an automorphism, $g(x_1 x_2) = (gx_1)(gx_2)$; and since $dg = (g, g)$, it follows that 8.1 holds. Thus any algebra is an algebra over the Hopf algebra of its automorphism group.

9. UNIVERSAL A -ALGEBRAS.

The foregoing examples of algebras over Hopf algebras arose naturally. We now show how to construct them in a wholesale fashion.

Let A be any Hopf algebra. It is easy to construct many modules over the algebra A (i.e. take quotients of A by left ideals, and then take direct sums of these). Let M be any graded A -module. Let M^n denote the tensor product of n copies of M . As in section 7, M^n is an A -module. Form the direct sum

$$T(M) = \sum_{n=0}^{\infty} M^n$$

where $M^0 = R$. Define $\mu: T(M) \otimes T(M) \rightarrow T(M)$ in terms of components $x \in M^r, y \in M^s$ by $\mu(x \otimes y) = x \otimes y \in M^{r+s}$ making use of the associative law $M^r \otimes M^s \approx M^{r+s}$. In this way $T(M)$ is an associative algebra. It is called the *free associative algebra* generated by M (also, the *tensor algebra* of M). Since the associative law $M^r \otimes M^s \approx M^{r+s}$ is an A -mapping, it follows that $T(M)$ is an algebra over the Hopf algebra A .

Form now the quotient of $T(M)$ by the ideal N generated by elements

$$(9.2) \quad x \otimes y - (-1)^{pq} y \otimes x \text{ where } x \in M_p, \quad y \in M_q.$$

The quotient, denoted by $U(M)$, is called the *free, commutative and associative algebra generated by M*. If we assume that the diagonal mapping Ψ of A is commutative, then it is readily verified that N is an A -submodule of $T(M)$. Hence $U(M)$ becomes an algebra over the Hopf algebra A .

As is well known, the algebra $T(M)$ is *universal* in the sense that any R -mapping of M into an algebra X extends to a unique mapping of algebras $T(M) \rightarrow X$. Furthermore, if X is an algebra over A , and $M \rightarrow X$ is an A -mapping, so also is $T(M) \rightarrow X$. A similar statement holds for $U(M)$ in case X is commutative.

Additional algebras over A can be constructed by taking a submodule of $T(M)$ or $U(M)$ forming the A -ideal it generates, and passing to the quotient algebra. It is easily seen that any A -algebra can be obtained as such a quotient.

In the special case where A is the algebra \mathcal{A}_p of reduced powers, only certain M 's are admissible, namely, those which satisfy the dimensionality restriction 4.9: $\mathcal{P}^i x = 0$ whenever $2i > \dim x$. Moreover, in forming $U(M)$, we must increase the ideal N so as to include all elements of the form

$$(9.3) \quad \mathcal{P}^k x - (x \otimes x \otimes \dots \otimes x) \text{ (} p \text{ factors)}, \quad x \in M_{2k}.$$

This insures that the relation 4.8, namely, $\mathcal{P}^k y = y^p$ is valid for $y \in U(M)_{2k}$. (It is a pleasant exercise in the use of the Adem-Cartan relations to show that N is an \mathcal{A}_p -module.) With these modifications, the resulting $U(M)$ is meaningful for algebraic topology.

10. REFORMULATION OF THE PROBLEM.

We are now in a position to formulate a problem similar to the one posed in section 2, but having a better chance of a positive solution. Recall that the algebra $F(R, q)^\infty$ of section 2 is small in that it has a single generator but is otherwise as big as