A trivial example is provided by any algebra X over R. Note first that $\varphi: R \otimes R \to R$ defined by $\varphi(r_1 \otimes r_2) = r_1 r_2$ is an isomorphism (recall that $\otimes = \otimes_R$). Set $\Psi = \varphi^{-1}: R \to R \otimes R$, then φ, Ψ give a natural structure of a Hopf algebra to the ground ring R. It is easily checked that the natural R-structure in $X \otimes X$ coincides with that defined by Ψ. Thus any algebra over the ground ring is an algebra over the ground ring regarded as a Hopf algebra.

As another example, let X be an algebra over R, and let π be a group of automorphisms of the algebra X. Let A be the group ring of π over R with the usual multiplication. Define the diagonal $\Psi: A \to A \otimes A$ to be the mapping induced by the diagonal mapping $d: \pi \to \pi \times \pi$. Then A becomes a Hopf algebra. Since any $g \in \pi$ is an automorphism, $g(x_1 x_2) = (gx_1)(gx_2)$; and since $dg = (g, g)$, it follows that 8.1 holds. Thus any algebra is an algebra over the Hopf algebra of its automorphism group.

9. Universal A-algebras.

The foregoing examples of algebras over Hopf algebras arose naturally. We now show how to construct them in a wholesale fashion.

Let A be any Hopf algebra. It is easy to construct many modules over the algebra A (i.e. take quotients of A by left ideals, and then take direct sums of these). Let M be any graded A-module. Let M^n denote the tensor product of n copies of M. As in section 7, M^n is an A-module. Form the direct sum

$$T(M) = \sum_{n=0}^{\infty} M^n$$

where $M^0 = R$. Define $\mu: T(M) \otimes T(M) \to T(M)$ in terms of components $x \in M^r$, $y \in M^s$ by $\mu(x \otimes y) = x \otimes y \in M^{r+s}$ making use of the associative law $M^r \otimes M^s \approx M^{r+s}$. In this way $T(M)$ is an associative algebra. It is called the *free associative algebra* generated by M (also, the *tensor algebra* of M). Since the associative law $M^r \otimes M^s \approx M^{r+s}$ is an A-mapping, it follows that $T(M)$ is an algebra over the Hopf algebra A.

Form now the quotient of $T(M)$ by the ideal N generated by elements

$$(9.2) \qquad x \otimes y - (-1)^{pq} y \otimes x \text{ where } x \in M_p, \quad y \in M_q .$$

The quotient, denoted by $U(M)$, is called the *free, commutative and associative algebra generated by* M. If we assume that the diagonal mapping Ψ of A is commutative, then it is readily verified that N is an A-submodule of $T(M)$. Hence $U(M)$ becomes an algebra over the Hopf algebra A.

As is well known, the algebra $T(M)$ is *universal* in the sense that any R-mapping of M into an algebra X extends to a unique mapping of algebras $T(M) \to X$. Furthermore, if X is an algebra over A, and $M \to X$ is an A-mapping, so also is $T(M) \to X$. A similar statement holds for $U(M)$ in case X is commutative.

Additional algebras over A can be constructed by taking a submodule of $T(M)$ or $U(M)$ forming the A-ideal it generates, and passing to the quotient algebra. It is easily seen that any A-algebra can be obtained as such a quotient.

In the special case where A is the algebra $\mathscr{A}_p$ of reduced powers, only certain M's are admissible, namely, those which satisfy the dimensionality restriction 4.9: $\mathscr{P}^i x = 0$ whenever $2i > \dim x$. Moreover, in forming $U(M)$, we must increase the ideal N so as to include all elements of the form

$$(9.3) \quad \mathscr{P}^k x - (x \otimes x \otimes \ldots \otimes x) \ (p \text{ factors}) , \quad x \in M_{2k} .$$

This insures that the relation 4.8, namely, $\mathscr{P}^k y = y^p$ is valid for $y \in U(M)_{2k}$. (It is a pleasant exercise in the use of the Adem-Cartan relations to show that N is an $\mathscr{A}_p$-module.) With these modifications, the resulting $U(M)$ is meaningful for algebraic topology.

10. Reformulation of the problem.

We are now in a position to formulate a problem similar to the one posed in section 2, but having a better chance of a positive solution. Recall that the algebra $F(R, q)^\infty$ of section 2 is small in that it has a single generator but is otherwise as big as