

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 7 (1961)  
**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE COHOMOLOGY ALGEBRA OF A SPACE  
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**Kapitel:** 8. Algebras over Hopf algebras.  
**DOI:** <https://doi.org/10.5169/seals-37129>

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leads to the formula

$$(\partial f)x = \partial(fx) + (-1)^{r+1}f(\partial x), \quad r = \deg f.$$

### 8. ALGEBRAS OVER HOPF ALGEBRAS.

We have seen that a graded algebra is a graded  $R$ -module  $X$  and an  $R$ -mapping  $\mu: X \otimes X \rightarrow X$ . Suppose now that  $X$  is also an  $A$ -module where  $A$  is a Hopf algebra over  $R$ . Then  $X \otimes X$  is an  $A$ -module as defined in section 7. We define  $X$  to be an algebra over the Hopf algebra  $A$  (briefly, an  $A$ -algebra) if the multiplication mapping  $\mu: X \otimes X \rightarrow X$  is an  $A$ -mapping.

In terms of elements  $a \in A$  and  $x_1, x_2 \in X$ , the condition for  $\mu$  to be an  $A$ -mapping takes the form

$$(8.1) \quad a(x_1 x_2) = \sum_i (-1)^{p q_i} (a'_i x_1) (a''_i x_2)$$

where

$$\Psi a = \sum_i a'_i \otimes a''_i, \quad p = \deg x_1, \quad q_i = \deg a''_i.$$

It is to be observed that this concept of an algebra over a Hopf algebra has arisen in a natural way. The discussion of section 7 demonstrates its inevitability. This being true there ought to be numerous examples.

The first, and for us the most important example, is the cohomology algebra of a space  $H^*(X; Z_p)$  over the Hopf algebra  $\mathcal{A}_p$  of reduced power operations. The cup-product formula

$$\mathcal{P}^k(x_1 x_2) = \sum_{i=0}^k (\mathcal{P}^i x_1) (\mathcal{P}^{k-i} x_2),$$

and the diagonal mapping  $\Psi \mathcal{P}^k = \sum_{i=0}^k \mathcal{P}^i \otimes \mathcal{P}^{k-i}$  show that 8.1 is satisfied.

Another example is provided by the differential, graded, augmented algebras of Cartan [8]. In this case,  $X$  is an augmented chain complex (i.e. a module over  $E(\partial, -1)$ , see § 7), and a chain mapping  $\mu: X \otimes X \rightarrow X$  defines an algebra structure in  $X$ .

A trivial example is provided by any algebra  $X$  over  $R$ . Note first that  $\varphi: R \otimes R \rightarrow R$  defined by  $\varphi(r_1 \otimes r_2) = r_1 r_2$  is an isomorphism (recall that  $\otimes = \otimes_R$ ). Set  $\Psi = \varphi^{-1}: R \rightarrow R \otimes R$ , then  $\varphi, \Psi$  give a natural structure of a Hopf algebra to the ground ring  $R$ . It is easily checked that the natural  $R$ -structure in  $X \otimes X$  coincides with that defined by  $\Psi$ . Thus any algebra over the ground ring is an algebra over the ground ring regarded as a Hopf algebra.

As another example, let  $X$  be an algebra over  $R$ , and let  $\pi$  be a group of automorphisms of the algebra  $X$ . Let  $A$  be the group ring of  $\pi$  over  $R$  with the usual multiplication. Define the diagonal  $\Psi: A \rightarrow A \otimes A$  to be the mapping induced by the diagonal mapping  $d: \pi \rightarrow \pi \times \pi$ . Then  $A$  becomes a Hopf algebra. Since any  $g \in \pi$  is an automorphism,  $g(x_1 x_2) = (gx_1)(gx_2)$ ; and since  $dg = (g, g)$ , it follows that 8.1 holds. Thus any algebra is an algebra over the Hopf algebra of its automorphism group.

## 9. UNIVERSAL $A$ -ALGEBRAS.

The foregoing examples of algebras over Hopf algebras arose naturally. We now show how to construct them in a wholesale fashion.

Let  $A$  be any Hopf algebra. It is easy to construct many modules over the algebra  $A$  (i.e. take quotients of  $A$  by left ideals, and then take direct sums of these). Let  $M$  be any graded  $A$ -module. Let  $M^n$  denote the tensor product of  $n$  copies of  $M$ . As in section 7,  $M^n$  is an  $A$ -module. Form the direct sum

$$T(M) = \sum_{n=0}^{\infty} M^n$$

where  $M^0 = R$ . Define  $\mu: T(M) \otimes T(M) \rightarrow T(M)$  in terms of components  $x \in M^r, y \in M^s$  by  $\mu(x \otimes y) = x \otimes y \in M^{r+s}$  making use of the associative law  $M^r \otimes M^s \approx M^{r+s}$ . In this way  $T(M)$  is an associative algebra. It is called the *free associative algebra* generated by  $M$  (also, the *tensor algebra* of  $M$ ). Since the associative law  $M^r \otimes M^s \approx M^{r+s}$  is an  $A$ -mapping, it follows that  $T(M)$  is an algebra over the Hopf algebra  $A$ .