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That Ψ is a homomorphism of algebras follows from 4.11. In this case Ψ is commutative; so the dual Hopf algebra $\mathscr{A}_p^*$ has a commutative multiplication. Milnor found an explicit and simple analysis of the structure of $\mathscr{A}_p^*$ as a tensor product of an exterior algebra and a polynomial algebra. Using an equally explicit form for the diagonal of $\mathscr{A}_p^*$, he was able to obtain results on the structure of $\mathscr{A}_p$ as an algebra. In particular, it is nilpotent.

It is to be emphasized that Hopf algebras have arisen in algebraic topology in these two very natural but quite different ways. This suggests that the concept is even more fundamental than had been thought. The next sections are devoted to developing the theme that Hopf algebras are basic because there are strong, purely algebraic reasons for introducing them.

7. Modules over Hopf algebras.

As a preliminary, let us review certain facts about the category $C(R)$ of graded modules over the ground ring R. The two functors $X \otimes Y$ and $\operatorname{Hom}(X, Y)$, where $\otimes$ and Hom are taken over R, have values in $C(R)$ when X, Y are in $C(R)$. The gradings of $X \otimes Y$ and $\operatorname{Hom}(X, Y)$ are defined by

$$(X \otimes Y)_r = \sum_{p+q=r} X_p \otimes Y_q$$

$$\operatorname{Hom}(X, Y)_r = \prod_{q-p=r} \operatorname{Hom}(X_p, Y_q) .$$

The index of the gradings ranges over all integers.

Furthermore, there are natural equivalences

$$(7.1) \qquad R \otimes X \approx X \approx X \otimes R\,, \quad \operatorname{Hom}(R, X) \approx X$$

obtained by identifying $r \otimes x = rx = x \otimes r$, and $f = f(1)$ for $f \in \operatorname{Hom}(R, X)$. The commutative law

$$(7.2) \qquad T\colon X \otimes Y \approx Y \otimes X$$

is a natural equivalence defined by $T(x \otimes y) = (-1)^{pq}\, y \otimes x$ where $x \in X_p$ and $y \in Y_q$. The associative law

$$(7.3) \qquad (X \otimes Y) \otimes Z \approx X \otimes (Y \otimes Z)$$

is a natural equivalence obtained by identifying $(x \otimes y) \otimes z$ with $x \otimes (y \otimes z)$.

There are three more associative laws involving $\otimes$ and Hom. The first is a natural equivalence

$$(7.4) \qquad U: \mathrm{Hom}(X \otimes Y, Z) \approx \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$$

defined by $((Uf)\,x)\,y = f\,(x \otimes y)$. The second is a natural transformation

$$(7.5) \qquad V: X \otimes \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(\mathrm{Hom}(X, Y), Z)$$

defined by $(V\,(x \otimes g))\,h = (-1)^{p(q+r)}\,g\,(h\,(x))$ where $p = \deg x$, $q = \deg g$, $r = \deg h$. In case each X_p is free and finitely generated, then V is an isomorphism. The third is a natural transformation

$$(7.6) \qquad W: \mathrm{Hom}(X, Y) \otimes Z \to \mathrm{Hom}(X, Y \otimes Z)$$

defined by $(W\,(h \otimes z))\,x = (-1)^{pq}\,(hx) \otimes z$ where $x \in X_p$, $z \in Z_q$ and $h \in \mathrm{Hom}(X, Y)$. If X or Z is free and finitely generated in each degree, then W is an isomorphism.

The fact that there are precisely four basic associative laws involving $\otimes$ and Hom may seem strange at first sight. But with a modest change of notation, the strangeness disappears. Write XY for $X \otimes Y$, and $X \diagdown Y$ for $\mathrm{Hom}(X, Y)$. Thinking of these operations as multiplication and division, the associative laws take on familiar forms, e.g. (7.4) becomes $(XY) \diagdown Z = X \diagdown (Y \diagdown Z)$. In case R is a field and everything is finitely generated, we can set $X^{-1} = \mathrm{Hom}(X, R)$, $\mathrm{Hom}(X, Y) = X^{-1} \otimes Y$; and then the analogy becomes a strict equivalence.

Now let A be a graded associative algebra over R with a unit, and let $C(A)$ be the category of A-modules and A-homomorphisms. Precisely, an object X of $C(A)$ is a graded R-module together with a multiplication $A \otimes X \to X$ (i.e. $A_p \otimes X_q \to X_{p+q}$ for all p, q) satisfying $a_1(a_2\,x) = (a_1\,a_2)\,x$ and $1x = x$. An A-homomorphism $f: X \to Y$ satisfies $f(ax) = af(x)$.

If $X, Y \in C(A)$, then $X \otimes Y$ is, in a natural way, an $(A \otimes A)$-module ($\otimes$ means $\otimes_R$);

$$(7.7) \qquad (a \otimes a')(x \otimes y) = (-1)^{qr}(ax) \otimes (a'y)\,, \qquad a' \in A_q\,, \quad x \in X_r\,.$$

The problem we shall consider is to give to $X \otimes Y$ the structure of an A-module so that the resulting tensor product is a functor of two variables from $C(A)$ to $C(A)$ such that the isomorphisms 7.1 to 7.3 are also in $C(A)$. Stated briefly, can we convert the tensor product to an internal operation in $C(A)$ so as to preserve standard properties ?

The answer is that each diagonal mapping $\Psi: A \to A \otimes A$ which makes A into a Hopf algebra converts the tensor product to an internal operation. In general, a homomorphism Ψ: $A \to B$ of algebras with unit defines a functor from $C(B)$ to $C(A)$ by the rule

$$A \otimes X \xrightarrow{\Psi \otimes 1} B \otimes X \to X \text{ for each } X \in C(B)\,.$$

Thus the condition for a Hopf algebra that $\Psi: A \to A \otimes A$ be a homomorphism of algebras follows naturally from this general principle.

If the isomorphism $R \otimes X \approx X$ of 7.1 is to be meaningful in $C(A)$, then R as well as X must be an A-module. This means a mapping $A \otimes R \to R$ of degree 0. Combining this with the natural isomorphism $A \approx A \otimes R$ yields a homomorphism $\varepsilon: A \to R$ of algebras with unit. Thus a realization of R in $C(A)$ coincides with an augmentation of A. Assume now that $R \otimes A \approx A \approx A \otimes R$ are A-mappings. It follows quickly that ε is a left and right unit for the coalgebra defined by Ψ. And this implies that $R \otimes X \approx X \approx X \otimes R$ are A-mappings for each $X \in C(A)$.

Let us assume now that 7.2 is an A-mapping in the special case $X = Y = A$. Since $\Psi a = a(1 \otimes 1)$, we have

$$T\Psi a = T(a(1 \otimes 1)) = aT(1 \otimes 1) = a(1 \otimes 1) = \Psi a\,.$$

Therefore Ψ is commutative; and this implies that 7.2 is an A-mapping for all $X, Y \in C(A)$.

Assume next that 7.3 is an A-mapping in the special case $X = Y = Z = A$. The statement "Ψ is a homomorphism of algebras " is easily seen to be equivalent to: " Ψ is an A-mapping ". Therefore

$$(1 \otimes \Psi)\Psi a = (1 \otimes \Psi) a (1 \otimes 1) = a(1 \otimes \Psi)(1 \otimes 1)$$
$$= a(1 \otimes (1 \otimes 1)) = a((1 \otimes 1) \otimes 1) = (\Psi \otimes 1)\Psi a .$$

It follows that Ψ is associative; and this implies that 7.3 is an A-mapping for all X, Y, $Z \in C(A)$.

We turn now to the functor Hom. If X, $Y \in C(A)$, then $\mathrm{Hom}(X, Y)$ is an $(A' \otimes A)$-module where A' denotes the opposite algebra of A. The action is given by

$$((a' \otimes a)f)x = (-1)^{q(r+s)} af(a'x)$$

where q, r, s are the degrees of a', a, f, respectively. Assume that A is a connected Hopf algebra, i.e. $A_0 \approx R$. By a theorem of Milnor and Moore [15], there is a unique isomorphism of Hopf algebras c: $A \approx A'$ which satisfies the identity $\varphi(c \otimes 1)\Psi = \eta\varepsilon$. It follows that $(c \otimes 1)\Psi$: $A \to A' \otimes A$ is a homomorphism of algebras with unit, thereby reducing $\mathrm{Hom}(X, Y)$ to an A-module. With no further assumptions on A, it can be verified (by tedious calculations) that each of the natural transformations 7.4, 7.5 and 7.6 are A-mappings for any X, Y, Z in $C(A)$.

To summarize, *a Hopf algebra structure in* A *is precisely what is needed to convert* $\otimes$ *and Hom to internal operations in* C(A) *with the customary properties.*

An important example of a category of modules over a Hopf algebra is the category of chain complexes and chain mappings. In this case the algebra A is the exterior algebra on one generator ∂ of degree -1, i.e. $A_0 = R$, $A_{-1} \approx R$ with ∂ as basis element, and $\partial\partial = 0$. A graded A-module is easily identified with the concept of chain complex, and A-mappings with chain mappings. In order that the tensor product of chain complexes shall have the usual A-structure, we must define Ψ by $\Psi\partial = \partial \otimes 1 + 1 \otimes \partial$. But this is the only choice which makes A a Hopf algebra.

In the literature, various combinations of signs have been used in defining the boundary operator in $\mathrm{Hom}(X, Y)$ where X, Y are chain complexes. The point of view of this section leads to the formula

$$(\partial f) x = \partial (fx) + (-1)^{r+1} f(\partial x) , \qquad r = \deg f .$$

8. Algebras over Hopf algebras.

We have seen that a graded algebra is a graded R-module X and an R-mapping $\mu \colon X \otimes X \to X$. Suppose now that X is also an A-module where A is a Hopf algebra over R. Then $X \otimes X$ is an A-module as defined in section 7. We define X to be *an algebra over the Hopf algebra* A (briefly, an A-algebra) if the multiplication mapping $\mu \colon X \otimes X \to X$ is an A-mapping.

In terms of elements $a \in A$ and $x_1, x_2 \in X$, the condition for μ to be an A-mapping takes the form

$$(8.1) \qquad a(x_1 x_2) = \textstyle\sum_i (-1)^{pq_i} (a_i' x_1)(a_i'' x_2)$$

where

$$\Psi a = \Sigma_i a_i' \otimes a_i'' , \qquad p = \deg x_1 , \qquad q_i = \deg a_i'' .$$

It is to be observed that this concept of an algebra over a Hopf algebra has arisen in a natural way. The discussion of section 7 demonstrates its inevitability. This being true there ought to be numerous examples.

The first, and for us the most important example, is the cohomology algebra of a space $H^*(X; Z_p)$ over the Hopf algebra $\mathscr{A}_p$ of reduced power operations. The cup-product formula

$$\mathscr{P}^k (x_1 x_2) = \sum_{i=0}^{k} (\mathscr{P}^i x_1)(\mathscr{P}^{k-i} x_2) ,$$

and the diagonal mapping $\Psi \mathscr{P}^k = \sum_{i=0}^{k} \mathscr{P}^i \otimes \mathscr{P}^{k-i}$ show that 8.1 is satisfied.

Another example is provided by the differential, graded, augmented algebras of Cartan [8]. In this case, X is an augmented chain complex (i.e. a module over $E(\partial, -1)$, see § 7), and a *chain* mapping $\mu \colon X \otimes X \to X$ defines an algebra structure in X.