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=2i(p-1); and the degree of a monomial in the generators is the sum of the degrees of the factors. After these definitions, it follows readily that, for each p, the cohomology  $H^*(X; \mathbb{Z}_p)$  of a space X is a graded  $\mathscr{A}_p$ -module.

As an abstract algebra,  $\mathcal{A}_p$  has a complicated structure. It is, of course, non-commutative. The Adem-Cartan relations give a kind of commutation law. A monomial in the generators

$$\beta^{\varepsilon_0} \mathscr{P}^{r_1} \beta^{\varepsilon_1} \mathscr{P}^{r_2} \dots \mathscr{P}^{r_k} \beta^{\varepsilon_k} \qquad (\varepsilon_j = 0 \text{ or } 1)$$

is called admissible if  $r_j \geq pr_{j+1} + \varepsilon_j$  for j = 1, 2, ..., k-1 and  $r_k \geq 1$ . The Adem-Cartan relations are rules for expressing inadmissible monomials in terms of admissible ones. Cartan has shown [9] that the admissible monomials form a vector space basis for  $\mathscr{A}_p$ . Thus there is a normal form for an element of  $\mathscr{A}_p$ .

Another consequence of the relations is the following result of Adem [3]:

4.12. The algebra  $\mathscr{A}_p$  is generated by  $\beta$  and the  $\mathscr{P}^{p^i}$  for i = 0, 1, 2, ...; and  $\mathscr{A}_2$  is generated by the  $\operatorname{Sq}^{2^i}$  for i = 0, 1, 2, ....

Let us see how this is proved for  $\mathscr{A}_2$ . Assume, inductively, that, for j < n, each  $\operatorname{Sq}^j$  is in the subalgebra generated by the  $\operatorname{Sq}^{2^i}$ . If n is not a power of 2, then  $n = a + 2^k$  where  $0 < a < 2^k$ . Set  $b = 2^k$  and apply 4.5. The coefficient in 4.5 of  $\operatorname{Sq}^{a+b} = \operatorname{Sq}^n$  is congruent to 1 mod 2. It follows that  $\operatorname{Sq}^n$  is decomposable as a sum of products of  $\operatorname{Sq}^j$  with j < n. The inductive hypothesis now implies that  $\operatorname{Sq}^n$  is in the subalgebra of the  $\operatorname{Sq}^{2^i}$ .

# 5. Non-realizability as cohomology algebras.

The preceding results will now be used to show that many of the graded algebras  $F(R, n)^h$  on one generator of dimension n and height h are not realizable. Recall that  $F(R, n)^2$  is realized by the n-sphere for each n and any ring R. So we shall restrict attention to the cases  $2 < h \le \infty$ .

First let  $R = \mathbb{Z}_2$ , and assume that  $F(\mathbb{Z}_2, n)^h$  is realized by a space X. Let  $x \in H^n(X; \mathbb{Z}_2)$  be the generator of  $H^*(X; \mathbb{Z}_2)$ . Since h > 2,  $x^2$  is not zero. By 4.3,  $\operatorname{Sq}^n x = x^2$  is not zero.

By 4.12,  $Sq^n$  is a sum of monomials in the  $Sq^{2^i}$  (i = 0, 1, 2, ...). This implies that  $Sq^{2^i}$  x is not zero for some  $2^i \le n$ . Its dimension  $n + 2^i$  is  $\le 2n$ . Since the groups  $H^q(X; \mathbb{Z}_2) = 0$  for n < q < 2n, it follows that  $2^i = n$ . This proves

5.1. If n is not a power of 2, and  $2 < h \le \infty$ , then  $F(Z_2, n)^h$  cannot be realized.

Now let p be a prime > 2, and consider  $F(Z_p, 2n)^h$ . Suppose it is realized by a space X for a certain n and h > p. Then the generator  $x \in H^{2n}(X; Z_p)$  is such that  $x^p$  is non-zero in  $H^{2np}(X; Z_p)$ . By 4.8,  $\mathscr{P}^n x = x^p$  is not zero. By 4.12,  $\mathscr{P}^n$  is a sum of monomials in the  $\mathscr{P}^{p^i}$  (i = 0, 1, 2, ...). It follows that some  $\mathscr{P}^{p^i} x \neq 0$  where  $p^i \leq n$ . Therefore the dimension  $2n + 2p^i (p-1)$  of  $\mathscr{P}^{p^i} x$  must coincide with one of the non-zero dimensions 2ns of  $H^*(X; Z_p)$ . Then

$$n(s-1) = p^{i}(p-1)$$
.

Since  $p^i \leq n$ , and n divides  $p^i$  (p-1), it follows that  $n = p^i m$  where m divides p-1. This proves

5.2. If n is not of the form  $p^i$  m where m divides p-1, and  $p < h \le \infty$ , then  $F(Z_p, 2n)^h$  cannot be realized.

Passing to integer coefficients, we shall derive the following complete result:

5.3. If  $3 < h \le \infty$ , then  $F(Z, 2n)^h$  is realizable if and only if n = 1 or 2.

We have seen in § 2 that  $F(Z, 2)^h$   $(F(Z, 4)^h)$  is realized by the complex (quaternionic) projective (h-1)-space. Conversely, suppose X realizes  $F(Z, 2n)^h$ . As  $H^*(X; Z)$  has no torsion, the universal coefficient theorem states that

$$H^*(X;Z) \otimes Z_p \approx H^*(X;Z_p) \; .$$

Since the reduction mod  $p: H^*(X; Z) \to H^*(X; Z_p)$  is a ring homomorphism, it follows that X realizes  $F(Z_p, 2n)^h$ . Taking p = 2, 5.1 asserts that  $2n = 2^s$  for some s. Taking p = 3, 5.2 asserts that  $n = 3^t$  or  $2.3^t$  for some t. Since both hold, we have  $2^{s-1} = 3^t$  or  $2.3^t$ . This implies t = 0, and therefore n = 1 or 2.

If we knew only that  $x^2 \neq 0$ , the above argument with p=2 shows that n is a power of 2. Therefore

5.4. If n is not a power of 2, then F(Z, 2n)<sup>3</sup> is not realizable.

Recall, by § 2, that  $F(Z, 8)^3$  and  $F(Z_p, 8)^3$  are realized by the Cayley projective plane. However, by 5.3,  $F(Z, 8)^4$  is not realizable. This is in accord with the fact that there is no projective 3-space over the Cayley numbers (due to non-associativity).

We turn next to the case of odd dimensional generators. Recall that  $F(Z, 2n + 1)^h$  is zero except for a Z in dimensions 0 and 2n + 1, and a  $Z_2$  in dimensions (2n + 1) k for 1 < k < h.

5.5. If  $2 < h \leq \infty$ , then  $F(Z, 1)^h$  is not realizable.

Assume X realizes  $F(Z, 1)^h$ . Let  $\eta: H^*(X; Z) \to H^*(X; Z_2)$  be reduction mod 2, and let  $x \in H^1(X; Z)$  be the generator. Then  $x^2$  is not zero and  $2x^2 = 0$ . It follows that  $\eta x$  and  $\eta(x^2) = (\eta x)^2$  are not zero. By 4.3 and 4.2,

$$(\eta x)^2 = \operatorname{Sq}^1 \eta x = \beta \eta x .$$

But  $\beta \eta$  is identically zero by the definition of  $\beta$ . This contradiction proves 5.5.

A second proof of 5.5 is based on the Hopf theorem that there exists a mapping  $f: X \to S^1$  (assuming X is a complex) such that  $x = f^* y$  where y generates  $H^1(S^1, Z)$ . Since  $y^2 = 0$ , it follows that  $x^2 = 0$ .

5.6.  $F(Z, 3)^3$  is realizable.

To see this, let Y be the suspension of the complex projective plane  $CP^2$ . If the latter is represented in the form  $S^2 \cup e_4$  (a 2-sphere with a 4-cell attached by the Hopf mapping  $S^3 \to S^2$ ), then  $Y = S^3 \cup e_5$  where  $e_5$  is attached by the suspension of the Hopf mapping. As this has order 2 in  $\pi_4(S^3)$ , the 5-cycle  $2e_5$  is spherical. Hence we may adjoin a 6-cell to Y obtaining a space  $X = S^3 \cup e_5 \cup e_6$  such that  $\partial e_6 = 2e_5$ . It is easily checked that  $H^*(X; Z)$  has Z in dimensions 0 and 3,  $Z_2$  in dimension 6, and is otherwise 0. We must show that the square of the

generator  $x \in H^3(X; \mathbb{Z})$  is non-zero in  $H^6(X; \mathbb{Z})$ . It is easily checked that the diagram

$$H^{3}(X;Z) \xrightarrow{\eta} H^{3}(X;Z_{2}) \xrightarrow{g} H^{3}(Y;Z_{2})$$

$$\downarrow f \quad \operatorname{Sq}^{3} \swarrow \operatorname{Sq}^{2} \qquad \downarrow \operatorname{Sq}^{2}$$

$$H^{6}(X;Z) \xrightarrow{\eta'} H^{6}(X;Z_{2}) \xleftarrow{\beta} H^{5}(X;Z_{2}) \xrightarrow{g'} H^{5}(Y;Z_{2})$$

is commutative where f is the squaring operation,  $\eta$  and  $\eta'$  are reduction mod 2, and g, g' are induced by the inclusion  $Y \subset X$ . The relation  $\beta Sq^2 = Sq^1 Sq^2 = Sq^3$  follows from 4.2, 4.5. All of the indicated groups except  $H^3(X; Z)$  are isomorphic to  $Z_2$ .

It follows that  $\eta$  is an epimorphism, and  $\eta'$  is an isomorphism. Since Y has the same 5-skeleton as X, g is an isomorphism and g' is a monomorphism. But both groups being  $Z_2$ , g' is an isomorphism. Since  $\partial e_6 = 2e_5$ , it follows that  $\beta$  is an isomorphism. Because  $\operatorname{Sq}^2$  commutes with suspension and is an isomorphism in  $CP^2$ , it gives an isomorphism in Y. Thus all the mappings of the diagram excepting f and  $\eta$  are isomorphisms. Since  $\eta$  is an epimorphism, commutativity implies that  $fx = x^2$  is not zero.

The preceding results are about as far as one can go using only the *primary* cohomology operations. There are secondary cohomology operations corresponding to the relations among the primary operations, and they are defined on a cohomology class on which certain primary operations are zero. The secondary operations have been exploited by J. F. Adams [1] to show that there are no mappings  $S^{2n-1} \to S^n$  of Hopf invariant 1 in cases other than n = 1, 2, 4 and 8. He proves this by showing that  $\operatorname{Sq}^{2i}$ , which is not decomposable in  $\mathscr{A}_2$ , is decomposable in terms of secondary operations for each  $i \geq 4$ . Using an argument similar to the proof of 5.1, Adams obtains the result

5.7. If  $i \ge 4$  and  $2 < h \le \infty$ , then  $F(Z_2, 2^i)^h$  is not realizable.

This and preceding results settle all cases for  $F(Z_2, n)^h$ . It is realizable precisely in the cases n = 1, 2, and 4 with  $3 \le h \le \infty$ , and n = 8 with h = 3.

The result of Adams has been extended to primes p > 2 by Liulevicius [13] and Shimada [17]. They have shown that  $\mathcal{P}^{p^i}$ 

is decomposable in terms of secondary operations for each  $i \ge 1$ . Using this result, 5.2 can be improved as follows:

5.8. If n is not a divisor of p — 1, and p < h  $\leq \infty$ , then  $F(Z_p, 2n)^h$  cannot be realized.

This leaves a good many unsettled cases. For example can  $F(Z_p, 2 (p-1))^3$  be realized for some p > 5? Can  $F(Z_5, 8)^4$  be realized? The cohomology of such a space would necessarily have torsion involving the prime 3. Likewise unsettled are the cases of  $F(Z, 2n+1)^h$  where n > 1, h > 2 and n = 1, h > 3. In view of the preceding results, it seems unlikely that any of these can be realized.

For a rough summary, let us exclude the trivial cases h=1,2. Then the only n's for which  $F(R,n)^h$  is known to be realizable are included among the integers 1, 2, 4 and 8. If  $R=Z,Z_2$ , or  $Z_3$  it is not realizable for any other n. If  $R=Z_p$ , it is not realizable for h>p and n>2 (p-1). In short,  $F(R,n)^h$  is not realizable except in rare cases involving small values of n or h.

These negative conclusions have interesting implications in algebra. The successful realizations were obtained by using projective spaces over the real numbers, complex numbers, quaternions, and Cayley numbers. If there is a real division algebra on n units, we can use it to realize  $F(Z_2, n)^3$ ; hence our non-existence results imply that n = 1, 2, 4 or 8. Again, since  $F(Z_3, 8)^4$  is not realizable, it follows that there is no real, associative division algebra on 8 units.

## 6. Hopf algebras.

Historically, we started with the preconception that the cohomology of a space is nothing more than a graded algebra, and we asked if certain simple graded algebras could be realized. On the whole we found that the answer was negative; and this was shown by using the fact that the algebra  $\mathscr{A}_p$  of reduced powers operates in  $H^*(X; Z_p)$ . Our preconception was misleading, the cohomology algebra of a space is something more than a graded algebra. Just how much more is not yet clear.