

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 7 (1961)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE COHOMOLOGY ALGEBRA OF A SPACE
Autor: Steenrod, N. E.
Kapitel: 5. NON-REALIZABILITY AS COHOMOLOGY ALGEBRAS
DOI: <https://doi.org/10.5169/seals-37129>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$= 2i(p - 1)$; and the degree of a monomial in the generators is the sum of the degrees of the factors. After these definitions, it follows readily that, for each p , the cohomology $H^*(X; Z_p)$ of a space X is a graded \mathcal{A}_p -module.

As an abstract algebra, \mathcal{A}_p has a complicated structure. It is, of course, non-commutative. The Adem-Cartan relations give a kind of commutation law. A monomial in the generators

$$\beta^{\varepsilon_0} \mathcal{P}^{r_1} \beta^{\varepsilon_1} \mathcal{P}^{r_2} \dots \mathcal{P}^{r_k} \beta^{\varepsilon_k} \quad (\varepsilon_j = 0 \text{ or } 1)$$

is called *admissible* if $r_j \geq pr_{j+1} + \varepsilon_j$ for $j = 1, 2, \dots, k-1$ and $r_k \geq 1$. The Adem-Cartan relations are rules for expressing inadmissible monomials in terms of admissible ones. Cartan has shown [9] that the admissible monomials form a vector space basis for \mathcal{A}_p . Thus there is a *normal form* for an element of \mathcal{A}_p .

Another consequence of the relations is the following result of Adem [3]:

4.12. *The algebra \mathcal{A}_p is generated by β and the \mathcal{P}^{p^i} for $i = 0, 1, 2, \dots$; and \mathcal{A}_2 is generated by the Sq^{2^i} for $i = 0, 1, 2, \dots$.*

Let us see how this is proved for \mathcal{A}_2 . Assume, inductively, that, for $j < n$, each Sq^j is in the subalgebra generated by the Sq^{2^i} . If n is not a power of 2, then $n = a + 2^k$ where $0 < a < 2^k$. Set $b = 2^k$ and apply 4.5. The coefficient in 4.5 of $\text{Sq}^{a+b} = \text{Sq}^n$ is congruent to 1 mod 2. It follows that Sq^n is decomposable as a sum of products of Sq^j with $j < n$. The inductive hypothesis now implies that Sq^n is in the subalgebra of the Sq^{2^i} .

5. NON-REALIZABILITY AS COHOMOLOGY ALGEBRAS.

The preceding results will now be used to show that many of the graded algebras $F(R, n)^h$ on one generator of dimension n and height h are not realizable. Recall that $F(R, n)^2$ is realized by the n -sphere for each n and any ring R . So we shall restrict attention to the cases $2 < h \leq \infty$.

First let $R = Z_2$, and assume that $F(Z_2, n)^h$ is realized by a space X . Let $x \in H^n(X; Z_2)$ be the generator of $H^*(X; Z_2)$. Since $h > 2$, x^2 is not zero. By 4.3, $\text{Sq}^n x = x^2$ is not zero.

By 4.12, Sq^n is a sum of monomials in the Sq^{2^i} ($i = 0, 1, 2, \dots$). This implies that $\text{Sq}^{2^i} x$ is not zero for some $2^i \leq n$. Its dimension $n + 2^i$ is $\leq 2n$. Since the groups $H^q(X; \mathbb{Z}_2) = 0$ for $n < q < 2n$, it follows that $2^i = n$. This proves

5.1. *If n is not a power of 2, and $2 < h \leq \infty$, then $F(\mathbb{Z}_2, n)^h$ cannot be realized.*

Now let p be a prime > 2 , and consider $F(\mathbb{Z}_p, 2n)^h$. Suppose it is realized by a space X for a certain n and $h > p$. Then the generator $x \in H^{2n}(X; \mathbb{Z}_p)$ is such that x^p is non-zero in $H^{2np}(X; \mathbb{Z}_p)$. By 4.8, $\mathcal{P}^n x = x^p$ is not zero. By 4.12, \mathcal{P}^n is a sum of monomials in the \mathcal{P}^{p^i} ($i = 0, 1, 2, \dots$). It follows that some $\mathcal{P}^{p^i} x \neq 0$ where $p^i \leq n$. Therefore the dimension $2n + 2p^i(p-1)$ of $\mathcal{P}^{p^i} x$ must coincide with one of the non-zero dimensions $2ns$ of $H^*(X; \mathbb{Z}_p)$. Then

$$n(s-1) = p^i(p-1).$$

Since $p^i \leq n$, and n divides $p^i(p-1)$, it follows that $n = p^i m$ where m divides $p-1$. This proves

5.2. *If n is not of the form $p^i m$ where m divides $p-1$, and $p < h \leq \infty$, then $F(\mathbb{Z}_p, 2n)^h$ cannot be realized.*

Passing to integer coefficients, we shall derive the following complete result:

5.3. *If $3 < h \leq \infty$, then $F(\mathbb{Z}, 2n)^h$ is realizable if and only if $n = 1$ or 2 .*

We have seen in § 2 that $F(\mathbb{Z}, 2)^h$ ($F(\mathbb{Z}, 4)^h$) is realized by the complex (quaternionic) projective $(h-1)$ -space. Conversely, suppose X realizes $F(\mathbb{Z}, 2n)^h$. As $H^*(X; \mathbb{Z})$ has no torsion, the universal coefficient theorem states that

$$H^*(X; \mathbb{Z}) \otimes \mathbb{Z}_p \approx H^*(X; \mathbb{Z}_p).$$

Since the reduction mod p : $H^*(X; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z}_p)$ is a ring homomorphism, it follows that X realizes $F(\mathbb{Z}_p, 2n)^h$. Taking $p = 2$, 5.1 asserts that $2n = 2^s$ for some s . Taking $p = 3$, 5.2 asserts that $n = 3^t$ or $2 \cdot 3^t$ for some t . Since both hold, we have $2^{s-1} = 3^t$ or $2 \cdot 3^t$. This implies $t = 0$, and therefore $n = 1$ or 2 .

If we knew only that $x^2 \neq 0$, the above argument with $p = 2$ shows that n is a power of 2. Therefore

5.4. *If n is not a power of 2, then $F(Z, 2n)^3$ is not realizable.*

Recall, by § 2, that $F(Z, 8)^3$ and $F(Z_p, 8)^3$ are realized by the Cayley projective plane. However, by 5.3, $F(Z, 8)^4$ is not realizable. This is in accord with the fact that there is no projective 3-space over the Cayley numbers (due to non-associativity).

We turn next to the case of odd dimensional generators. Recall that $F(Z, 2n+1)^h$ is zero except for a Z in dimensions 0 and $2n+1$, and a Z_2 in dimensions $(2n+1)k$ for $1 < k < h$.

5.5. *If $2 < h \leq \infty$, then $F(Z, 1)^h$ is not realizable.*

Assume X realizes $F(Z, 1)^h$. Let $\eta: H^*(X; Z) \rightarrow H^*(X; Z_2)$ be reduction mod 2, and let $x \in H^1(X; Z)$ be the generator. Then x^2 is not zero and $2x^2 = 0$. It follows that ηx and $\eta(x^2) = (\eta x)^2$ are not zero. By 4.3 and 4.2,

$$(\eta x)^2 = \text{Sq}^1 \eta x = \beta \eta x .$$

But $\beta \eta$ is identically zero by the definition of β . This contradiction proves 5.5.

A second proof of 5.5 is based on the Hopf theorem that there exists a mapping $f: X \rightarrow S^1$ (assuming X is a complex) such that $x = f^* y$ where y generates $H^1(S^1, Z)$. Since $y^2 = 0$, it follows that $x^2 = 0$.

5.6. *$F(Z, 3)^3$ is realizable.*

To see this, let Y be the suspension of the complex projective plane CP^2 . If the latter is represented in the form $S^2 \cup e_4$ (a 2-sphere with a 4-cell attached by the Hopf mapping $S^3 \rightarrow S^2$), then $Y = S^3 \cup e_5$ where e_5 is attached by the suspension of the Hopf mapping. As this has order 2 in $\pi_4(S^3)$, the 5-cycle $2e_5$ is spherical. Hence we may adjoin a 6-cell to Y obtaining a space $X = S^3 \cup e_5 \cup e_6$ such that $\partial e_6 = 2e_5$. It is easily checked that $H^*(X; Z)$ has Z in dimensions 0 and 3, Z_2 in dimension 6, and is otherwise 0. We must show that the square of the

generator $x \in H^3(X; \mathbb{Z})$ is non-zero in $H^6(X; \mathbb{Z})$. It is easily checked that the diagram

$$\begin{array}{ccccc}
 H^3(X; \mathbb{Z}) & \xrightarrow{\eta} & H^3(X; \mathbb{Z}_2) & \xrightarrow{g} & H^3(Y; \mathbb{Z}_2) \\
 \downarrow f & \text{Sq}^3 \swarrow & \searrow \text{Sq}^2 & & \downarrow \text{Sq}^2 \\
 H^6(X; \mathbb{Z}) & \xrightarrow{\eta'} & H^6(X; \mathbb{Z}_2) & \xleftarrow{\beta} & H^5(X; \mathbb{Z}_2) \xrightarrow{g'} H^5(Y; \mathbb{Z}_2)
 \end{array}$$

is commutative where f is the squaring operation, η and η' are reduction mod 2, and g, g' are induced by the inclusion $Y \subset X$. The relation $\beta \text{Sq}^2 = \text{Sq}^1 \text{Sq}^2 = \text{Sq}^3$ follows from 4.2, 4.5. All of the indicated groups except $H^3(X; \mathbb{Z})$ are isomorphic to \mathbb{Z}_2 .

It follows that η is an epimorphism, and η' is an isomorphism. Since Y has the same 5-skeleton as X , g is an isomorphism and g' is a monomorphism. But both groups being \mathbb{Z}_2 , g' is an isomorphism. Since $\delta e_6 = 2e_5$, it follows that β is an isomorphism. Because Sq^2 commutes with suspension and is an isomorphism in CP^2 , it gives an isomorphism in Y . Thus all the mappings of the diagram excepting f and η are isomorphisms. Since η is an epimorphism, commutativity implies that $fx = x^2$ is not zero.

The preceding results are about as far as one can go using only the *primary* cohomology operations. There are secondary cohomology operations corresponding to the relations among the primary operations, and they are defined on a cohomology class on which certain primary operations are zero. The secondary operations have been exploited by J. F. Adams [1] to show that there are no mappings $S^{2n-1} \rightarrow S^n$ of Hopf invariant 1 in cases other than $n = 1, 2, 4$ and 8 . He proves this by showing that Sq^{2i} , which is not decomposable in \mathcal{A}_2 , is decomposable in terms of secondary operations for each $i \geq 4$. Using an argument similar to the proof of 5.1, Adams obtains the result

5.7. *If $i \geq 4$ and $2 < h \leq \infty$, then $F(\mathbb{Z}_2, 2^i)^h$ is not realizable.*

This and preceding results settle all cases for $F(\mathbb{Z}_2, n)^h$. It is realizable precisely in the cases $n = 1, 2$, and 4 with $3 \leq h \leq \infty$, and $n = 8$ with $h = 3$.

The result of Adams has been extended to primes $p > 2$ by Liulevicius [13] and Shimada [17]. They have shown that \mathcal{P}^{p^i}

is decomposable in terms of secondary operations for each $i \geq 1$. Using this result, 5.2 can be improved as follows:

5.8. *If n is not a divisor of $p - 1$, and $p < h \leq \infty$, then $F(Z_p, 2n)^h$ cannot be realized.*

This leaves a good many unsettled cases. For example can $F(Z_p, 2(p - 1))^3$ be realized for some $p > 5$? Can $F(Z_5, 8)^4$ be realized? The cohomology of such a space would necessarily have torsion involving the prime 3. Likewise unsettled are the cases of $F(Z, 2n + 1)^h$ where $n > 1$, $h > 2$ and $n = 1$, $h > 3$. In view of the preceding results, it seems unlikely that any of these can be realized.

For a rough summary, let us exclude the trivial cases $h = 1, 2$. Then the only n 's for which $F(R, n)^h$ is known to be realizable are included among the integers 1, 2, 4 and 8. If $R = Z, Z_2$, or Z_3 it is not realizable for any other n . If $R = Z_p$, it is not realizable for $h > p$ and $n > 2(p - 1)$. In short, $F(R, n)^h$ is not realizable except in rare cases involving small values of n or h .

These negative conclusions have interesting implications in algebra. The successful realizations were obtained by using projective spaces over the real numbers, complex numbers, quaternions, and Cayley numbers. If there is a real division algebra on n units, we can use it to realize $F(Z_2, n)^3$; hence our non-existence results imply that $n = 1, 2, 4$ or 8. Again, since $F(Z_3, 8)^4$ is not realizable, it follows that there is no real, associative division algebra on 8 units.

6. HOPF ALGEBRAS.

Historically, we started with the preconception that the cohomology of a space is nothing more than a graded algebra, and we asked if certain simple graded algebras could be realized. On the whole we found that the answer was negative; and this was shown by using the fact that the algebra \mathcal{A}_p of reduced powers operates in $H^*(X; Z_p)$. Our preconception was misleading, the cohomology algebra of a space is something more than a graded algebra. Just how much more is not yet clear.