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THE COHOMOLOGY ALGEBRA OF A SPACE [1)]

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1. Introduction.

The history of the development of the concept of the cohomology algebra of a space is marked by quite a few wrong turns, blind alleys, and fallacious preconceptions. The purpose of this article is to trace this development with emphasis on the errors that were made. This may be useful since the story is not yet complete, and the final form of the concept is still to be determined.

The key to the existence of the multiplication of cohomology classes lay in the Alexander duality theorem, especially in the form it was given by Pontrjagin in 1934: If X is a closed subset of the n-sphere S^n, then, for any coefficient group G, the singular homology group $H_{n-q-1}(S^n - X; G)$ is isomorphic to char H_q $(X; \text{char } G)$ where char means character group, and H_q is Čech homology. This latter group coincides with what we now call the cohomology group, and we denote it by $H^q(X; G)$. Since $H_i(S^n) = 0$ for $0 < i < n$, it follows by exactness that

$$H^q(X; G) \approx H_{n-q-1}(S^n - X; G) \approx H_{n-q}(S^n, S^n - X; G)$$

for $0 < q < n - 1$, the latter isomorphism being given by the boundary operator. If G is a ring R, the relative groups $H_r(S^n, S^n - X; R)$ admit an intersection theory in the sense of Lefschetz, because S^n is a manifold. This induces a multiplication

$$H^p(X; R) \otimes H^q(X; R) \to H^{p+q}(X; R) . \tag{1.1}$$

To obtain a fully satisfactory duality theorem, it was necessary to show that this multiplication is independent of the imbedding and of n. During the years 1935 to 1938 this was achieved in papers by Alexander, Čech, Gordon and Whitney. A direct

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1) Talk delivered at the Zurich Colloquium on Differential Geometry and Topology, June 1960.

internal description was given of the groups H^q, and of the multiplication 1.1. Their constructions were triumphs of ingenuity, and it was puzzling that something so basic should be so difficult. The difficulty was dissipated by Lefschetz in 1942 when he showed that the multiplication is the composition of the cross-product with values in $H^{p+q}(X\times X; R)$ followed by the homomorphism induced by the diagonal mapping $X\rightarrow X\times X$.

The upshot was a readily defined product, easily proved to be associative, to have a unit in dimension 0, and to satisfy the commutation law

$$(1.2) \qquad xy = (-1)^{pq}xy\ , \qquad x\in H^p,\ y\in H^q\ .$$

With this success there was great hope that the multiplication would lead to deeper insight into the topology of spaces. To give this hope a precise formulation, one spoke of the *cohomology algebra* $H^*(X; R)$. It was defined to be the direct sum $\Sigma_{q=0}^{\infty} H^q(X; R)$ with the multiplication determined by the multiplications of the component parts. It is clearly an associative algebra with unit. And now one can ask if the theory of algebras can be brought to bear on topological problems via $H^*(X; R)$.

The above definition of H^* constituted a wrong turn into a blind alley. It is an error which has not yet been fully erased. The mistake lay in forming the direct sum over the dimensional index q. No one has yet found a valid geometric reason for adding cohomology classes of different dimensions. The only algebraic reason for doing so was to make H^* into a familiar algebraic object. This was a forcing of the mathematics into a preconceived pattern. There was no gain in doing so. For example, the interesting part of the algebra is the sum $\Sigma_{q>0} H^q$; but if X is a finite complex, this part lies in the radical of H^*. Unfortunately, algebraic theory has little to say about the radical. Its major results concern the quotient by the radical. Even worse, H^* is badly non-commutative in spite of the fact that the rule 1.2 for commuting two elements is just as useful as the commutative law $xy = yx$.

It has now come to be recognized that the proper algebraic concept, to which the cohomology of a space conforms, is that of a graded algebra. A *graded algebra* A is, first of all, a sequence

$\{A^q\}$, $q = 0, 1, 2, \ldots$, of R-modules. Thus, an element of A is an element of some A^q, and q is called its degree. Elements may be added only if they have the same degree. In addition, homomorphisms $A^p \otimes A^q \to A^{p+q}$ are given for all $p, q \geqq 0$. These define a bilinear product xy for all $x, y \in A$. The product is required to be associative.

An ordinary algebra C is converted into a graded algebra A by setting $A^0 = C$ and $A^q = 0$ for $q > 0$. In this way, a graded algebra is a *generalization* of the notion of an algebra. Thus we are free to generalize the properties of algebras to graded algebras in any convenient manner which conforms in degree 0.

In particular, a graded algebra A is called commutative if $xy = (-1)^{pq}\, yx$ for all $x \in A^p$ and $y \in A^q$. Thus, what was once called the *anti*-commutative law is now called the commutative law. And the cohomology algebra of a space is an associative, commutative, graded algebra.

A unit of a graded algebra A is an element $1 \in A^0$ such that $1x = x = x1$ for all $x \in A$. An augmentation ε of A is a homomorphism $\varepsilon : A \to R$ of graded algebras with unit. Thus $\varepsilon\,(A^q) = 0$ for $q > 0$, and $\varepsilon\,(1) = 1$. In case ε gives an isomorphism in degree 0, $A^0 \approx R$, then A is called *connected.*

If P is a space consisting of a single point, it is clear that $H^*(P; R) = R$ as a graded algebra. For any space X, the mapping $\eta : X \to P$ induces a monomorphism $\eta^* : R \to H^*(X; R)$ and $\eta^*\,(1)$ is the unique unit of $H^*(X; R)$. Finally, any mapping $P \to X$ induces an augmentation of $H^*(X; R)$. Clearly X is arcwise connected if and only if $H^*(X; R)$ is connected; and then the augmentation is unique.

2. Realizing a graded algebra as a cohomology algebra.

Let Z denote the ring of integers. It is well known that, if B is a graded Z-module such that $B^0 = Z$, B^1 is free, and B^n is finitely generated for each n, then there is a space X which realizes B in that $H^*(X; Z) \approx B$. One solves this problem, for a single dimension n, by a cluster C_n of n-spheres and $(n+1)$-cells; and then the general case is solved by a union of the C_n's with a common point. If Z is replaced by another of the simpler ground rings, there are no serious difficulties in solving the realization problem.

Suppose however that B is a graded, commutative, associative and connected algebra over R. A *realization* of B is a space X such that $H^*(X; R) \approx B$ as graded algebras. The problem of deciding when a given B is realizable has not been solved, and is very difficult. To make the problem precise, we shall use singular cohomology groups, and require X to be a CW-complex. A natural attack on the problem is to consider first the case of certain simple B's, and then pass to more complicated ones.

Let $F(R, n)^\infty$ denote the graded, free, commutative, associative, and connected algebra over R on one generator x of dimension n; and let $F(R, n)^h$ be the quotient algebra obtained by setting $x^h = 0$ (h = height of x). Thus, if n is even, $F(R, n)^\infty$ is the " polynomial " algebra of x, i.e. the monomials $1 = x^0, x^1, \ldots, x^k, \ldots$ form a module basis; and it is a free R-module. If n is odd, the commutative law demands that $2x^2 = 0$, so $2x^k = 0$ for all $k \geqq 2$. Thus the kn dimensional part is isomorphic to $R/2R$ for $k \geqq 2$. Setting $x^h = 0$ replaces all component groups in dimensions $\geqq hn$ by zero.

We will discuss the problem of realizing $F(R, n)^h$ in the special cases where R is the ring Z of integers, or the field Z_p of integers reduced modulo a prime p. First, we will list three trivial cases.

The space consisting of a single point realizes $F(R, n)^1$ for all R and n.

The n-sphere S^n realizes $F(R, n)^2$ for all R and n.

If n is odd, S^n also realizes $F(R, n)^h$ for all $2 \leqq h \leqq \infty$ providing $R/2R = 0$, because the relation $2x^2 = 0$ must then imply that $x^2 = 0$; so h is effectively 2. The condition $R/2R = 0$ holds for $R = Z_p$, $p > 2$.

The projective spaces over the real numbers, complex numbers and quaternions provide realizations of quite a few of the F's. Consider first the real projective n-space P^n. Taking $R = Z_2$, $H_q(P^n) \approx Z_2$ for $0 \leqq q \leqq n$, and the non-trivial element is represented by any subspace $P^q \subset P^n$. Now any P^q

can be made the intersection of $n - q$ projective $(n - 1)$-subspaces. The duality between intersections of cycles and products of cocycles shows that $H^*(P^n; Z_2) \approx F(Z_2, 1)^{n+1}$. Let P^∞ be the union of a sequence $P^0 \subset P^1 \subset \ldots \subset P^n \subset \ldots$. Then P^∞ realizes $F(Z_2, 1)^\infty$.

The complex projective n-space CP^n has real dimension $2n$. It has no torsion, its odd dimensional Betti numbers are zero, and its Betti number is 1 in each even dimension $\leqq 2n$. A generator of $H_{2q}(CP^n; R)$ is provided by any projective subspace $CP^q \subset CP^n$. Again, the duality between intersections and products shows that $H^*(CP^n; R) \approx F(R, 2)^{n+1}$ for any R. Forming CP^∞, as above, realizes $F(R, 2)^\infty$.

The quaternionic projective n-space QP^n has real dimension $4n$, no torsion, and non-zero Betti numbers equal to 1 in dimensions $4q \leqq 4n$. A similar argument shows that QP^n realizes $F(R, 4)^{n+1}$ for each R and each $n \leqq \infty$.

The Cayley numbers (on 8 units) is non-associative. As a result the usual notion of the equivalence of two sets of homogeneous coordinates fails to be transitive; hence there is no Cayley projective n-space. An exception is $n = 2$, because any two Cayley numbers generate an associative subalgebra. Using this, Hopf [11] constructed a Cayley projective plane M of real dimension 16. It has no torsion, and its non-zero Betti numbers are equal to 1 in dimensions 0, 8 and 16. An appropriate argument shows that M realizes $F(R, 8)^3$ for any R.

The preceding results are very encouraging, a great many of the F's are realized by spaces which are not too complicated. One might be led by these to expect that any F can be realized. A bit of ingenuity in putting spaces together should do the trick. Once the case of one generator is thus solved, the special case of many generators given by a tensor product of F's for various n's and h's, can be solved by cartesian products of the separate realizations. Thus it begins to appear likely that any graded, commutative and associative algebra can be realized.

The historical fact is that topologists were lulled to sleep by the above considerations. Their preconception of the nature of the cohomology algebra appeared to be justified. However they were awakened abruptly in 1952 by the result of Adem [2] which states: If n is not a power of 2, and $3 \leqq h \leqq \infty$, then $F(Z_2, n)^h$ cannot be realized.

Subsequent revelations showed that the situation is even worse: the preceding examples of realizations of F's are nearly all that exist. The method for proving this uses the fact that the cyclic reduced p$^{\text{th}}$ powers, which operate in the algebra $H^*(X; Z_p)$, satisfy certain relations. In the next three sections we will discuss these operations, and their implications for the realization problem.

3. Construction of the squaring operations.

Before presenting the algebra of the reduced power operations, it may be worthwhile to give a recently improved form of the definition of the operations themselves. For simplicity we restrict ourselves to the case of the prime 2.

Let π be a cyclic group of order 2 with generator T ($T^2 = 1$). Let W be an acyclic complex on which π acts freely. Algebraically, W is a free resolution of Z over π. Geometrically, W can be taken to be the union of an infinite sequence of spheres $S^0 \subset S^1 \subset \ldots \subset S^n \subset \ldots$ where each is the equator of its successor, and T is the antipodal transformation. Identifying equivalent points under π gives the infinite dimensional real projective space P with W as its 2-fold covering. Recall that $H^*(P; Z_2)$ is the polynomial ring $F(Z_2, 1)^\infty$ on the one dimensional generator U.

Let K be any space, form the cartesian product $W \times K \times K$, and let π act in this space by $T(w, x, y) = (Tw, y, x)$. Then T has no fixed points. Identifying equivalent points gives a space, denoted by $W \times_\pi K^2$, which is covered twice by $W \times K^2$. Imbed $W \times K$ in $W \times K^2$ by $(w, x) \to (w, x, x)$. Then π transforms $W \times K$ into itself with $T(w, x) = (Tw, x)$. It follows that $W \times K$ covers a subspace $P \times K$ imbedded in $W \times_\pi K^2$. This gives the diagram

$$(3.1) \qquad P \times K \xrightarrow{i} W \times_\pi K^2 \xleftarrow{h} W \times K^2 \xrightarrow{g} K^2$$

where i is the inclusion, h is the covering, and g is the obvious projection.

The squaring operations will be defined by the following diagram of cohomology with coefficients Z_2:

$$(3.2) \qquad H^q(K) \xrightarrow{\Psi} H^{2q}(W \times_\pi K^2) \xrightarrow{i^*} H^{2q}(P \times K) = \sum_{j=0}^{2q} H^{2q-j}(P) \otimes H^j(K) .$$

The function Ψ is still to be defined. The equality on the right is the standard decomposition of the Künneth theorem $H^*(P \times K) = H^*(P) \otimes H^*(K)$.

To define Ψ, recall that h defines an isomorphism

$$H^*(W \times_\pi K^2) \approx H^*_\pi(W \times K^2)$$

where H^*_π denotes the cohomology of $W \times K^2$ based on cochains which are invariant under the action of π. This isomorphism was studied first by Eilenberg [10] who called it the *equivariant* cohomology. To define Ψ, it suffices therefore to define $\Psi' \colon H^q(K) \to H^{2q}_\pi(W \times K^2)$. For simplicity, assume K is a cell complex, and that $W \times K^2$ has as cells the products of cells of its factors. Then g in 3.1 is a cellular mapping. Let u_1 be a q-cocycle representing $u \in H^q(K)$. Then $u_1 \otimes u_1$ is an invariant $2q$-cocycle of $K \times K$ where π acts by $T(x, y) = (y, x)$. Since g is cellular, it induces a cochain mapping $g^{\#}$. Since $gT = Tg$, it follows that $g^{\#}(u_1 \otimes u_1)$ is an invariant cocycle, and it thereby represents an element $\Psi' u$ in $H^{2q}_\pi(W \times K^2)$. The fact that the π-cohomology class of $g^{\#}(u_1 \otimes u_1)$ depends only on the class of u_1 can be proved using Lemma 5.2 in [19]. This completes the definition of Ψ' and hence of Ψ.

If $x \in \mathrm{H}^q(K)$, by 3.2 the composition $i^* \Psi x$ decomposes into a sum. Since $H^*(P)$ is the polynomial ring in U, this sum has the form $\Sigma_j U^{2q-j} \otimes v_j$ where $v_j \in H^j(K)$ is a uniquely defined function of x. It can be shown that, for $j < q$, each $v_j = 0$. The remaining v_j are called the *reduced squares* of x. Thus

$$(3.3) \qquad i^* \Psi x = \sum_{i=0}^{q} U^{q-i} \otimes \mathrm{Sq}^i x .$$

The advantage of this definition is that it analyzes the previous definition in terms of two standard operations (the i^*

and the Künneth formula) and the single new operation Ψ. This simplifies the derivation of the properties of the Sq^i, and illuminates their origin.

Note that the projection $W \times_\pi K^2 \to P$ is a fibration with fibre K^2. For each $x \in H^q(K)$, $x \otimes x$ is a cohomology class of the fibre. The element Ψx is a canonical extension of $x \otimes x$ to a class on the total space.

4. The algebras of reduced power operations.

The definition of the reduced powers, given above for complexes, extends to the Čech cohomology of general spaces by taking direct limits of the operations in the nerves of coverings. The extension to the singular theory, by the method of acyclic models, has been carried through by Araki [4].

The main property of the squares is that

$$\mathrm{Sq}^i\colon H^q(X;Z_2) \to H^{q+i}(X;Z_2)$$

is a homomorphism for each space X and each $i \geqq 0$, and if $f\colon X \to Y$ is a mapping, Sq^i commutes with the induced homomorphism f^* of cohomology. The principal algebraic properties are

(4.1) $\mathrm{Sq}^0 = \text{identity}.$

(4.2) $\mathrm{Sq}^1 =$ the Bockstein operator β of the coefficient sequence

$$0 \to Z_2 \to Z_4 \to Z_2 \to 0 .$$

(4.3) If $\dim x = n$, then $\mathrm{Sq}^n x = x^2$.

(4.4) If $\dim x = n$, then $\mathrm{Sq}^i x = 0$ for all $i > n$.

(4.5) The Adem relations [2]: If $a < 2b$, then

$$\mathrm{Sq}^a\,\mathrm{Sq}^b = \binom{b-1}{a}\mathrm{Sq}^{a+b} + \sum_{j=1}^{[a/2]}\binom{b-j-1}{a-2j}\mathrm{Sq}^{a+b-j}\,\mathrm{Sq}^j .$$

(4.6) The Cartan formula [6]: If $x, y \in H^*(X;Z_2)$, then

$$\mathrm{Sq}^i(xy) = \sum_{j=0}^{i}(\mathrm{Sq}^j x)(\mathrm{Sq}^{i-j} y) .$$

The generalization of the reduced powers to primes $p > 2$ takes on a somewhat unexpected form. Many of the terms in the formula corresponding to 3.3 prove to be zero. The remaining terms can be expressed using a sequence of homomorphisms

$$\mathscr{P}^i_p \colon H^q(X; Z_p) \to H^{q+2i(p-1)}(X; Z_p)\,, \qquad i = 0, 1, 2, \ldots$$

and the Bockstein operator β of the coefficient sequence

$$0 \to Z_p \to Z_{p^2} \to Z_p \to 0\,.$$

(The analogy with $p = 2$ associates $\mathscr{P}^i_2$ with Sq^{2i}.) Their algebraic properties are

(4.7) $\qquad \mathscr{P}^0 = \text{identity}.$

(4.8) If $\dim x = 2n$, then $\mathscr{P}^n x = x^p$.

(4.9) If $2i > \dim x$, then $\mathscr{P}^i x = 0$.

(4.10) The Adem-Cartan relations [3, 9]: If $a < pb$, then

$$\mathscr{P}^a \mathscr{P}^b = \sum_{i=0}^{[a/p]} (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-ip} \mathscr{P}^{a+b-i} \mathscr{P}^i\,.$$

If $a < pb + 1$, then

$$\mathscr{P}^a \beta \mathscr{P}^b = \sum_{i=0}^{[a/p]} (-1)^{a+i} \binom{(p-1)(b-i)}{a-ip} \beta \mathscr{P}^{a+b-i} \mathscr{P}^i$$
$$+ \sum_{i=0}^{[a-1/p]} (-1)^{a-1+i} \binom{(p-1)(b-i)-1}{a-ip-1} \mathscr{P}^{a+b-i} \beta \mathscr{P}^i\,.$$

(4.11) The Cartan formula [18]: If $x, y \in H^*(X; Z_p)$, then

$$\mathscr{P}^i(xy) = \sum_{j=0}^{i} (\mathscr{P}^j x)(\mathscr{P}^{i-j} y)\,, \qquad \beta(xy) = (\beta x) y + (-1)^{\dim x} x (\beta y)\,.$$

The algebra $\mathscr{A}_2$ of the squaring operations is defined to be the graded associative algebra over Z_2 generated by the Sq^i ($i = 0, 1, 2, \ldots$) subject to the relations 4.1 and 4.5. Similarly, for $p > 2$, $\mathscr{A}_p$ is the graded associative algebra over Z_p generated by β and the $\mathscr{P}^i$ subject to the relations 4.7, 4.10 and $\beta^2 = 0$. Degrees are defined by $\deg(\mathrm{Sq}^i) = i$, $\deg(\beta) = 1$, $\deg(\mathscr{P}^i)$

$= 2i(p-1)$; and the degree of a monomial in the generators is the sum of the degrees of the factors. After these definitions, it follows readily that, for each p, the cohomology $H^*(X; Z_p)$ of a space X is a graded $\mathscr{A}_p$-module.

As an abstract algebra, $\mathscr{A}_p$ has a complicated structure. It is, of course, non-commutative. The Adem-Cartan relations give a kind of commutation law. A monomial in the generators

$$\beta^{\varepsilon_0} \mathscr{P}^{r_1} \beta^{\varepsilon_1} \mathscr{P}^{r_2} \ldots \mathscr{P}^{r_k} \beta^{\varepsilon_k} \qquad (\varepsilon_j = 0 \quad \text{or} \quad 1)$$

is called *admissible* if $r_j \geqq p r_{j+1} + \varepsilon_j$ for $j = 1, 2, \ldots, k-1$ and $r_k \geqq 1$. The Adem-Cartan relations are rules for expressing inadmissible monomials in terms of admissible ones. Cartan has shown [9] that the admissible monomials form a vector space basis for $\mathscr{A}_p$. Thus there is a *normal form* for an element of $\mathscr{A}_p$.

Another consequence of the relations is the following result of Adem [3]:

4.12. *The algebra $\mathscr{A}_p$ is generated by β and the $\mathscr{P}^{p^i}$ for* $i = 0, 1, 2, \ldots$; *and $\mathscr{A}_2$ is generated by the* Sq^{2^i} *for* $i = 0, 1, 2, \ldots$.

Let us see how this is proved for $\mathscr{A}_2$. Assume, inductively, that, for $j < n$, each Sq^j is in the subalgebra generated by the Sq^{2^i}. If n is not a power of 2, then $n = a + 2^k$ where $0 < a < 2^k$. Set $b = 2^k$ and apply 4.5. The coefficient in 4.5 of $\mathrm{Sq}^{a+b} = \mathrm{Sq}^n$ is congruent to 1 mod 2. It follows that Sq^n is decomposable as a sum of products of Sq^j with $j < n$. The inductive hypothesis now implies that Sq^n is in the subalgebra of the Sq^{2^i}.

5. Non-realizability as cohomology algebras.

The preceding results will now be used to show that many of the graded algebras $F(R, n)^h$ on one generator of dimension n and height h are not realizable. Recall that $F(R, n)^2$ is realized by the n-sphere for each n and any ring R. So we shall restrict attention to the cases $2 < h \leqq \infty$.

First let $R = Z_2$, and assume that $F(Z_2, n)^h$ is realized by a space X. Let $x \in H^n(X; Z_2)$ be the generator of $H^*(X; Z_2)$. Since $h > 2$, x^2 is not zero. By 4.3, $\mathrm{Sq}^n x = x^2$ is not zero.

By 4.12, Sq^n is a sum of monomials in the Sq^{2^i} $(i = 0, 1, 2, \ldots)$. This implies that $\mathrm{Sq}^{2^i} x$ is not zero for some $2^i \leqq n$. Its dimension $n + 2^i$ is $\leqq 2n$. Since the groups $H^q(X; Z_2) = 0$ for $n < q < 2n$, it follows that $2^i = n$. This proves

5.1. *If* n *is not a power of* 2, *and* $2 < \mathrm{h} \leqq \infty$, *then* $\mathrm{F}(\mathrm{Z}_2, \mathrm{n})^h$ *cannot be realized.*

Now let p be a prime > 2, and consider $F(Z_p, 2n)^h$. Suppose it is realized by a space X for a certain n and $h > p$. Then the generator $x \in H^{2n}(X; Z_p)$ is such that x^p is non-zero in $H^{2np}(X; Z_p)$. By 4.8, $\mathcal{P}^n x = x^p$ is not zero. By 4.12, $\mathcal{P}^n$ is a sum of monomials in the $\mathcal{P}^{p^i}$ $(i = 0, 1, 2, \ldots)$. It follows that some $\mathcal{P}^{p^i} x \neq 0$ where $p^i \leqq n$. Therefore the dimension $2n + 2p^i(p - 1)$ of $\mathcal{P}^{p^i} x$ must coincide with one of the non-zero dimensions $2ns$ of $H^*(X; Z_p)$. Then

$$n(s-1) = p^i(p-1) .$$

Since $p^i \leqq n$, and n divides $p^i(p - 1)$, it follows that $n = p^i m$ where m divides $p - 1$. This proves

5.2. *If* n *is not of the form* $\mathrm{p}^i \mathrm{m}$ *where* m *divides* $\mathrm{p} - 1$, *and* $\mathrm{p} < \mathrm{h} \leqq \infty$, then $\mathrm{F}(\mathrm{Z}_p, 2\mathrm{n})^h$ *cannot be realized.*

Passing to integer coefficients, we shall derive the following complete result:

5.3. *If* $3 < \mathrm{h} \leqq \infty$, *then* $\mathrm{F}(\mathrm{Z}, 2\mathrm{n})^h$ *is realizable if and only if* $\mathrm{n} = 1$ *or* 2.

We have seen in § 2 that $F(Z, 2)^h$ $(F(Z, 4)^h)$ is realized by the complex (quaternionic) projective $(h - 1)$-space. Conversely, suppose X realizes $F(Z, 2n)^h$. As $H^*(X; Z)$ has no torsion, the universal coefficient theorem states that

$$H^*(X; Z) \otimes Z_p \approx H^*(X; Z_p) .$$

Since the reduction mod p: $H^*(X; Z) \to H^*(X; Z_p)$ is a ring homomorphism, it follows that X realizes $F(Z_p, 2n)^h$. Taking $p = 2$, 5.1 asserts that $2n = 2^s$ for some s. Taking $p = 3$, 5.2 asserts that $n = 3^t$ or 2.3^t for some t. Since both hold, we have $2^{s-1} = 3^t$ or 2.3^t. This implies $t = 0$, and therefore $n = 1$ or 2.

If we knew only that $x^2 \neq 0$, the above argument with $p = 2$ shows that n is a power of 2. Therefore

5.4. *If* n *is not a power of* 2, *then* $F(Z, 2n)^3$ *is not realizable.*

Recall, by § 2, that $F(Z, 8)^3$ and $F(Z_p, 8)^3$ are realized by the Cayley projective plane. However, by 5.3, $F(Z, 8)^4$ is not realizable. This is in accord with the fact that there is no projective 3-space over the Cayley numbers (due to non-associativity).

We turn next to the case of odd dimensional generators. Recall that $F(Z, 2n+1)^h$ is zero except for a Z in dimensions 0 and $2n+1$, and a Z_2 in dimensions $(2n+1)k$ for $1 < k < h$.

5.5. *If* $2 < h \leqq \infty$, *then* $F(Z, 1)^h$ *is not realizable.*

Assume X realizes $F(Z, 1)^h$. Let η: $H^*(X; Z) \to H^*(X; Z_2)$ be reduction mod 2, and let $x \in H^1(X; Z)$ be the generator. Then x^2 is not zero and $2x^2 = 0$. It follows that ηx and $\eta(x^2) = (\eta x)^2$ are not zero. By 4.3 and 4.2,

$$(\eta x)^2 = \mathrm{Sq}^1 \eta x = \beta \eta x .$$

But $\beta\eta$ is identically zero by the definition of β. This contradiction proves 5.5.

A second proof of 5.5 is based on the Hopf theorem that there exists a mapping f: $X \to S^1$ (assuming X is a complex) such that $x = f^* y$ where y generates $H^1(S^1, Z)$. Since $y^2 = 0$, it follows that $x^2 = 0$.

5.6. $F(Z, 3)^3$ *is realizable.*

To see this, let Y be the suspension of the complex projective plane CP^2. If the latter is represented in the form $S^2 \cup e_4$ (a 2-sphere with a 4-cell attached by the Hopf mapping $S^3 \to S^2$), then $Y = S^3 \cup e_5$ where e_5 is attached by the suspension of the Hopf mapping. As this has order 2 in $\pi_4(S^3)$, the 5-cycle $2e_5$ is spherical. Hence we may adjoin a 6-cell to Y obtaining a space $X = S^3 \cup e_5 \cup e_6$ such that $\partial e_6 = 2e_5$. It is easily checked that $H^*(X; Z)$ has Z in dimensions 0 and 3, Z_2 in dimension 6, and is otherwise 0. We must show that the square of the generator $x \in H^3(X; Z)$ is non-zero in $H^6(X; Z)$. It is easily checked that the diagram

$$\begin{tikzcd} H^3(X;Z) \arrow[r, "\eta"] \arrow[d, "f"] & H^3(X;Z_2) \arrow[rr, "g"] \arrow[dl, "Sq^3"'] \arrow[dr, "Sq^2"] & & H^3(Y;Z_2) \arrow[d, "Sq^2"] \\ H^6(X;Z) \arrow[r, "\eta'"] & H^6(X;Z_2) & H^5(X;Z_2) \arrow[l, "\beta"'] \arrow[r, "g'"] & H^5(Y;Z_2) \end{tikzcd}$$

is commutative where f is the squaring operation, η and η' are reduction mod 2, and g, g' are induced by the inclusion $Y \subset X$. The relation $\beta Sq^2 = Sq^1 Sq^2 = Sq^3$ follows from 4.2, 4.5. All of the indicated groups except $H^3(X; Z)$ are isomorphic to Z_2.

It follows that η is an epimorphism, and η' is an isomorphism. Since Y has the same 5-skeleton as X, g is an isomorphism and g' is a monomorphism. But both groups being Z_2, g' is an isomorphism. Since $\partial e_6 = 2e_5$, it follows that β is an isomorphism. Because Sq^2 commutes with suspension and is an isomorphism in CP^2, it gives an isomorphism in Y. Thus all the mappings of the diagram excepting f and η are isomorphisms. Since η is an epimorphism, commutativity implies that $fx = x^2$ is not zero.

The preceding results are about as far as one can go using only the *primary* cohomology operations. There are secondary cohomology operations corresponding to the relations among the primary operations, and they are defined on a cohomology class on which certain primary operations are zero. The secondary operations have been exploited by J. F. Adams [1] to show that there are no mappings $S^{2n-1} \to S^n$ of Hopf invariant 1 in cases other than $n = 1, 2, 4$ and 8. He proves this by showing that Sq^{2^i}, which is not decomposable in $\mathscr{A}_2$, is decomposable in terms of secondary operations for each $i \geqq 4$. Using an argument similar to the proof of 5.1, Adams obtains the result

5.7. *If* $i \geqq 4$ *and* $2 < h \leqq \infty$, *then* $F(Z_2, 2^i)^h$ *is not realizable.*

This and preceding results settle all cases for $F(Z_2, n)^h$. It is realizable precisely in the cases $n = 1, 2,$ and 4 with $3 \leqq h \leqq \infty$, and $n = 8$ with $h = 3$.

The result of Adams has been extended to primes $p > 2$ by Liulevicius [13] and Shimada [17]. They have shown that $\mathscr{P}^{p^i}$

is decomposable in terms of secondary operations for each $i \geqq 1$. Using this result, 5.2 can be improved as follows:

5.8. *If* n *is not a divisor of* $\mathrm{p}-1$, *and* $\mathrm{p} < \mathrm{h} \leqq \infty$, *then* $\mathrm{F}(\mathrm{Z}_p, 2\mathrm{n})^h$ *cannot be realized.*

This leaves a good many unsettled cases. For example can $F(Z_p, 2(p-1))^3$ be realized for some $p > 5$? Can $F(Z_5, 8)^4$ be realized? The cohomology of such a space would necessarily have torsion involving the prime 3. Likewise unsettled are the cases of $F(Z, 2n+1)^h$ where $n > 1$, $h > 2$ and $n = 1$, $h > 3$. In view of the preceding results, it seems unlikely that any of these can be realized.

For a rough summary, let us exclude the trivial cases $h = 1, 2$. Then the only n's for which $F(R, n)^h$ is known to be realizable are included among the integers 1, 2, 4 and 8. If $R = Z, Z_2$, or Z_3 it is not realizable for any other n. If $R = Z_p$, it is not realizable for $h > p$ and $n > 2(p-1)$. In short, $F(R, n)^h$ is not realizable except in rare cases involving small values of n or h.

These negative conclusions have interesting implications in algebra. The successful realizations were obtained by using projective spaces over the real numbers, complex numbers, quaternions, and Cayley numbers. If there is a real division algebra on n units, we can use it to realize $F(Z_2, n)^3$; hence our non-existence results imply that $n = 1, 2, 4$ or 8. Again, since $F(Z_3, 8)^4$ is not realizable, it follows that there is no real, associative division algebra on 8 units.

6. Hopf algebras.

Historically, we started with the preconception that the cohomology of a space is nothing more than a graded algebra, and we asked if certain simple graded algebras could be realized. On the whole we found that the answer was negative; and this was shown by using the fact that the algebra $\mathscr{A}_p$ of reduced powers operates in $H^*(X; Z_p)$. Our preconception was misleading, the cohomology algebra of a space is something more than a graded algebra. Just how much more is not yet clear.

However a certain part of this additional structure can be clarified; and we shall do so in this and subsequent sections.

Let us recall the concept of a Hopf algebra A. In the first place A is a graded, associative algebra over the ground ring R with a unit and an augmentation $\varepsilon: A \to R$. The unit is regarded as a homomorphism of algebras $\eta: R \to A$ defined by $\eta(1_R) = 1_A$. Define $A \otimes A$ to be the graded module whose component of degree r is given by

$$(A \otimes A)_r = \sum_{q=0}^{r} A_q \otimes A_{r-q} \,.$$

The multiplication mappings $A_p \otimes A_q \to A_{p+q}$ are the components of a mapping $\varphi: A \otimes A \to A$ of graded R-modules. Define an algebra structure in $A \otimes A$ by

$$(a \otimes b)(a' \otimes b') = (-1)^{qr}(aa') \otimes (bb')$$

where $q = \deg b$, and $r = \deg a'$. The final element of structure is a " diagonal mapping "

$$\Psi: A \to A \otimes A$$

which is required to be a homomorphism of algebras with unit, and to satisfy the conditions

$$(\varepsilon \otimes 1)\,\Psi a = 1 \otimes a\,, \qquad (1 \otimes \varepsilon)\,\Psi a = a \otimes 1$$

as mappings $A \to R \otimes A$, and $A \to A \otimes R$.

Furthermore, Ψ is usually required to be *associative*, i.e. the mappings $(1 \otimes \Psi)\Psi$ and $(\Psi \otimes 1)\Psi$ of A into $A \otimes A \otimes A$ coincide. Sometimes Ψ is required to be *commutative*, i.e. $T\Psi = \Psi$ where $T: A \otimes A \to A \otimes A$ is defined by $T(a \otimes b) = (-1)^{pq}\, b \otimes a$ where $p = \deg a$, $q = \deg b$. In most applications, φ or Ψ is commutative, but rarely both.

The Hopf algebra structure thereby consists of the mappings

$$A \xrightarrow{\Psi} A \otimes A \xrightarrow{\varphi} A\,, \qquad R \xrightarrow{\eta} A \xrightarrow{\varepsilon} R\,.$$

The asymmetry in Ψ, φ and η, ε gives rise to a duality. The graded module A together with a mapping $\Psi: A \to A \otimes A$ is called a *coalgebra* and $\varepsilon: A \to R$ is called a unit for the coalgebra. The requirement that Ψ be a homomorphism of algebras is equivalent to demanding that φ be a homomorphism of coalgebras. This compatibility of φ, Ψ is expressed in a neutral fashion by requiring that the following diagram be commutative:

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\varphi} & A \\ \downarrow \Psi\otimes\Psi & & \\ A\otimes A\otimes A\otimes A & & \Big\downarrow \Psi \\ \searrow 1\otimes T\otimes 1 & & \\ A\otimes A\otimes A\otimes A & \xrightarrow{\varphi\otimes\varphi} & A\otimes A \end{array}$$

The concept of Hopf algebra arose first in Hopf's study [12] of the homology of a group manifold G. The diagonal mapping and the multiplication mapping

$$G \xrightarrow{\Psi} G\times G \xrightarrow{\varphi} G$$

induce homomorphisms of homology over a field of coefficients

$$H_*(G) \xrightarrow{\Psi_*} H_*(G)\otimes H_*(G) \xrightarrow{\varphi_*} H_*(G)$$

and the group homomorphisms $1\to G\to 1$ induce the unit and augmentation. In this case Ψ_* is commutative. If, instead, we pass to cohomology, then φ^* becomes the diagonal mapping, and the multiplication Ψ^* is commutative.

Because of this application to Lie groups, Hopf algebras have been studied extensively. One of the best results, due to Borel [5], assumes that R is a perfect field of characteristic p and A has a commutative multiplication $A_0 \approx R$ and A_q is of finite rank for each q. The conclusion is that, as an algebra, A is a tensor product of subalgebras B^1, B^2, ... each on a single generator b_1, b_2, ... If $p>2$ and deg b_i is odd, B^i is an exterior algebra ($b_i^2=0$); and if $p=2$, or if $p>2$ and deg b_i is even, B^i is either the polynomial ring on b_i, or the polynomial ring truncated by the relation $b_i^h=0$ where h is a power of p.

It was Milnor [14] who observed that the reduced power algebra $\mathscr{A}_p$ is a Hopf algebra with the diagonal mapping defined by

$$\Psi\mathscr{P}^k = \Sigma_{i=0}^k \mathscr{P}^i\otimes\mathscr{P}^{k-i}\,, \qquad \Psi\beta = \beta\otimes 1 + 1\otimes\beta\,.$$

That Ψ is a homomorphism of algebras follows from 4.11. In this case Ψ is commutative; so the dual Hopf algebra $\mathscr{A}_p^*$ has a commutative multiplication. Milnor found an explicit and simple analysis of the structure of $\mathscr{A}_p^*$ as a tensor product of an exterior algebra and a polynomial algebra. Using an equally explicit form for the diagonal of $\mathscr{A}_p^*$, he was able to obtain results on the structure of $\mathscr{A}_p$ as an algebra. In particular, it is nilpotent.

It is to be emphasized that Hopf algebras have arisen in algebraic topology in these two very natural but quite different ways. This suggests that the concept is even more fundamental than had been thought. The next sections are devoted to developing the theme that Hopf algebras are basic because there are strong, purely algebraic reasons for introducing them.

7. Modules over Hopf algebras.

As a preliminary, let us review certain facts about the category $C(R)$ of graded modules over the ground ring R. The two functors $X \otimes Y$ and $\operatorname{Hom}(X, Y)$, where $\otimes$ and Hom are taken over R, have values in $C(R)$ when X, Y are in $C(R)$. The gradings of $X \otimes Y$ and $\operatorname{Hom}(X, Y)$ are defined by

$$(X \otimes Y)_r = \sum_{p+q=r} X_p \otimes Y_q$$

$$\operatorname{Hom}(X, Y)_r = \prod_{q-p=r} \operatorname{Hom}(X_p, Y_q) .$$

The index of the gradings ranges over all integers.

Furthermore, there are natural equivalences

$$(7.1) \qquad R \otimes X \approx X \approx X \otimes R\,, \quad \operatorname{Hom}(R, X) \approx X$$

obtained by identifying $r \otimes x = rx = x \otimes r$, and $f = f(1)$ for $f \in \operatorname{Hom}(R, X)$. The commutative law

$$(7.2) \qquad T\colon X \otimes Y \approx Y \otimes X$$

is a natural equivalence defined by $T(x \otimes y) = (-1)^{pq}\, y \otimes x$ where $x \in X_p$ and $y \in Y_q$. The associative law

$$(7.3) \qquad (X \otimes Y) \otimes Z \approx X \otimes (Y \otimes Z)$$

is a natural equivalence obtained by identifying $(x \otimes y) \otimes z$ with $x \otimes (y \otimes z)$.

There are three more associative laws involving $\otimes$ and Hom. The first is a natural equivalence

$$(7.4) \qquad U\colon \operatorname{Hom}(X \otimes Y, Z) \approx \operatorname{Hom}(X, \operatorname{Hom}(Y, Z))$$

defined by $((Uf)\,x)\,y = f\,(x \otimes y)$. The second is a natural transformation

$$(7.5) \qquad V : X \otimes \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(\operatorname{Hom}(X, Y), Z)$$

defined by $(V\,(x \otimes g))\,h = (-1)^{p(q+r)}\,g\,(h\,(x))$ where $p = \deg x$, $q = \deg g$, $r = \deg h$. In case each X_p is free and finitely generated, then V is an isomorphism. The third is a natural transformation

$$(7.6) \qquad W\colon \operatorname{Hom}(X, Y) \otimes Z \to \operatorname{Hom}(X, Y \otimes Z)$$

defined by $(W\,(h \otimes z))\,x = (-1)^{pq}\,(hx) \otimes z$ where $x \in X_p$, $z \in Z_q$ and $h \in \operatorname{Hom}(X, Y)$. If X or Z is free and finitely generated in each degree, then W is an isomorphism.

The fact that there are precisely four basic associative laws involving $\otimes$ and Hom may seem strange at first sight. But with a modest change of notation, the strangeness disappears. Write XY for $X \otimes Y$, and $X \backslash Y$ for $\operatorname{Hom}(X, Y)$. Thinking of these operations as multiplication and division, the associative laws take on familiar forms, e.g. (7.4) becomes $(XY)\backslash Z = X\backslash(Y\backslash Z)$. In case R is a field and everything is finitely generated, we can set $X^{-1} = \operatorname{Hom}(X, R)$, $\operatorname{Hom}(X, Y) = X^{-1} \otimes Y$; and then the analogy becomes a strict equivalence.

Now let A be a graded associative algebra over R with a unit, and let $C(A)$ be the category of A-modules and A-homomorphisms. Precisely, an object X of $C(A)$ is a graded R-module together with a multiplication $A \otimes X \to X$ (i.e. $A_p \otimes X_q \to X_{p+q}$ for all p, q) satisfying $a_1(a_2\,x) = (a_1\,a_2)\,x$ and $1x = x$. An A-homomorphism $f\colon X \to Y$ satisfies $f(ax) = af(x)$.

If $X, Y \in C(A)$, then $X \otimes Y$ is, in a natural way, an $(A \otimes A)$-module ($\otimes$ means $\otimes_R$);

$$(7.7) \qquad (a \otimes a')(x \otimes y) = (-1)^{qr}(ax) \otimes (a'y)\,, \qquad a' \in A_q\,, \quad x \in X_r\,.$$

The problem we shall consider is to give to $X \otimes Y$ the structure of an A-module so that the resulting tensor product is a functor of two variables from $C(A)$ to $C(A)$ such that the isomorphisms 7.1 to 7.3 are also in $C(A)$. Stated briefly, can we convert the tensor product to an internal operation in $C(A)$ so as to preserve standard properties ?

The answer is that each diagonal mapping $\Psi: A \to A \otimes A$ which makes A into a Hopf algebra converts the tensor product to an internal operation. In general, a homomorphism Ψ: $A \to B$ of algebras with unit defines a functor from $C(B)$ to $C(A)$ by the rule

$$A \otimes X \xrightarrow{\Psi \otimes 1} B \otimes X \to X \text{ for each } X \in C(B)\,.$$

Thus the condition for a Hopf algebra that $\Psi: A \to A \otimes A$ be a homomorphism of algebras follows naturally from this general principle.

If the isomorphism $R \otimes X \approx X$ of 7.1 is to be meaningful in $C(A)$, then R as well as X must be an A-module. This means a mapping $A \otimes R \to R$ of degree 0. Combining this with the natural isomorphism $A \approx A \otimes R$ yields a homomorphism $\varepsilon: A \to R$ of algebras with unit. Thus a realization of R in $C(A)$ coincides with an augmentation of A. Assume now that $R \otimes A \approx A \approx A \otimes R$ are A-mappings. It follows quickly that ε is a left and right unit for the coalgebra defined by Ψ. And this implies that $R \otimes X \approx X \approx X \otimes R$ are A-mappings for each $X \in C(A)$.

Let us assume now that 7.2 is an A-mapping in the special case $X = Y = A$. Since $\Psi a = a\,(1 \otimes 1)$, we have

$$T\Psi a = T(a\,(1 \otimes 1)) = aT(1 \otimes 1) = a\,(1 \otimes 1) = \Psi a\,.$$

Therefore Ψ is commutative; and this implies that 7.2 is an A-mapping for all X, $Y \in C(A)$.

Assume next that 7.3 is an A-mapping in the special case $X = Y = Z = A$. The statement " Ψ is a homomorphism of algebras " is easily seen to be equivalent to: " Ψ is an A-mapping ". Therefore

$$\begin{aligned}(1 \otimes \Psi)\Psi a &= (1 \otimes \Psi) a (1 \otimes 1) = a (1 \otimes \Psi)(1 \otimes 1) \\ &= a (1 \otimes (1 \otimes 1)) = a ((1 \otimes 1) \otimes 1) = (\Psi \otimes 1) \Psi a .\end{aligned}$$

It follows that Ψ is associative; and this implies that 7.3 is an A-mapping for all $X, Y, Z \in C(A)$.

We turn now to the functor Hom. If $X, Y \in C(A)$, then $\mathrm{Hom}(X, Y)$ is an $(A' \otimes A)$-module where A' denotes the opposite algebra of A. The action is given by

$$((a' \otimes a) f) x = (-1)^{q(r+s)} a f(a' x)$$

where q, r, s are the degrees of a', a, f, respectively. Assume that A is a connected Hopf algebra, i.e. $A_0 \approx R$. By a theorem of Milnor and Moore [15], there is a unique isomorphism of Hopf algebras $c\colon A \approx A'$ which satisfies the identity $\varphi (c \otimes 1) \Psi = \eta\varepsilon$. It follows that $(c \otimes 1)\Psi\colon A \to A' \otimes A$ is a homomorphism of algebras with unit, thereby reducing $\mathrm{Hom}(X, Y)$ to an A-module. With no further assumptions on A, it can be verified (by tedious calculations) that each of the natural transformations 7.4, 7.5 and 7.6 are A-mappings for any X, Y, Z in $C(A)$.

To summarize, *a Hopf algebra structure in* A *is precisely what is needed to convert* $\otimes$ *and Hom to internal operations in* C(A) *with the customary properties.*

An important example of a category of modules over a Hopf algebra is the category of chain complexes and chain mappings. In this case the algebra A is the exterior algebra on one generator ∂ of degree -1, i.e. $A_0 = R$, $A_{-1} \approx R$ with ∂ as basis element, and $\partial\partial = 0$. A graded A-module is easily identified with the concept of chain complex, and A-mappings with chain mappings. In order that the tensor product of chain complexes shall have the usual A-structure, we must define Ψ by $\Psi\partial = \partial \otimes 1 + 1 \otimes \partial$. But this is the only choice which makes A a Hopf algebra.

In the literature, various combinations of signs have been used in defining the boundary operator in $\mathrm{Hom}(X, Y)$ where X, Y are chain complexes. The point of view of this section leads to the formula

$$(\partial f)x = \partial(fx) + (-1)^{r+1} f(\partial x), \quad r = \deg f.$$

8. Algebras over Hopf algebras.

We have seen that a graded algebra is a graded R-module X and an R-mapping $\mu\colon X \otimes X \to X$. Suppose now that X is also an A-module where A is a Hopf algebra over R. Then $X \otimes X$ is an A-module as defined in section 7. We define X to be *an algebra over the Hopf algebra* A (briefly, an A-algebra) if the multiplication mapping $\mu\colon X \otimes X \to X$ is an A-mapping.

In terms of elements $a \in A$ and $x_1, x_2 \in X$, the condition for μ to be an A-mapping takes the form

$$(8.1) \qquad a(x_1 x_2) = \textstyle\sum_i (-1)^{pq_i} (a_i' x_1)(a_i'' x_2)$$

where

$$\Psi a = \Sigma_i a_i' \otimes a_i'', \quad p = \deg x_1, \quad q_i = \deg a_i''.$$

It is to be observed that this concept of an algebra over a Hopf algebra has arisen in a natural way. The discussion of section 7 demonstrates its inevitability. This being true there ought to be numerous examples.

The first, and for us the most important example, is the cohomology algebra of a space $H^*(X; Z_p)$ over the Hopf algebra $\mathscr{A}_p$ of reduced power operations. The cup-product formula

$$\mathscr{P}^k(x_1 x_2) = \sum_{i=0}^{k} (\mathscr{P}^i x_1)(\mathscr{P}^{k-i} x_2),$$

and the diagonal mapping $\Psi \mathscr{P}^k = \sum_{i=0}^{k} \mathscr{P}^i \otimes \mathscr{P}^{k-i}$ show that 8.1 is satisfied.

Another example is provided by the differential, graded, augmented algebras of Cartan [8]. In this case, X is an augmented chain complex (i.e. a module over $E(\partial, -1)$, see § 7), and a *chain* mapping $\mu\colon X \otimes X \to X$ defines an algebra structure in X.

A trivial example is provided by any algebra X over R. Note first that $\varphi: R \otimes R \to R$ defined by $\varphi(r_1 \otimes r_2) = r_1 r_2$ is an isomorphism (recall that $\otimes = \otimes_R$). Set $\Psi = \varphi^{-1}: R \to R \otimes R$, then φ, Ψ give a natural structure of a Hopf algebra to the ground ring R. It is easily checked that the natural R-structure in $X \otimes X$ coincides with that defined by Ψ. Thus any algebra over the ground ring is an algebra over the ground ring regarded as a Hopf algebra.

As another example, let X be an algebra over R, and let π be a group of automorphisms of the algebra X. Let A be the group ring of π over R with the usual multiplication. Define the diagonal $\Psi: A \to A \otimes A$ to be the mapping induced by the diagonal mapping $d: \pi \to \pi \times \pi$. Then A becomes a Hopf algebra. Since any $g \in \pi$ is an automorphism, $g(x_1 x_2) = (gx_1)(gx_2)$; and since $dg = (g, g)$, it follows that 8.1 holds. Thus any algebra is an algebra over the Hopf algebra of its automorphism group.

9. Universal A-algebras.

The foregoing examples of algebras over Hopf algebras arose naturally. We now show how to construct them in a wholesale fashion.

Let A be any Hopf algebra. It is easy to construct many modules over the algebra A (i.e. take quotients of A by left ideals, and then take direct sums of these). Let M be any graded A-module. Let M^n denote the tensor product of n copies of M. As in section 7, M^n is an A-module. Form the direct sum

$$T(M) = \sum_{n=0}^{\infty} M^n$$

where $M^0 = R$. Define $\mu: T(M) \otimes T(M) \to T(M)$ in terms of components $x \in M^r$, $y \in M^s$ by $\mu(x \otimes y) = x \otimes y \in M^{r+s}$ making use of the associative law $M^r \otimes M^s \approx M^{r+s}$. In this way $T(M)$ is an associative algebra. It is called the *free associative algebra* generated by M (also, the *tensor algebra* of M). Since the associative law $M^r \otimes M^s \approx M^{r+s}$ is an A-mapping, it follows that $T(M)$ is an algebra over the Hopf algebra A.

Form now the quotient of $T(M)$ by the ideal N generated by elements

$$(9.2) \qquad x \otimes y - (-1)^{pq} y \otimes x \text{ where } x \in M_p , \quad y \in M_q .$$

The quotient, denoted by $U(M)$, is called the *free, commutative and associative algebra generated by* M. If we assume that the diagonal mapping Ψ of A is commutative, then it is readily verified that N is an A-submodule of $T(M)$. Hence $U(M)$ becomes an algebra over the Hopf algebra A.

As is well known, the algebra $T(M)$ is *universal* in the sense that any R-mapping of M into an algebra X extends to a unique mapping of algebras $T(M) \to X$. Furthermore, if X is an algebra over A, and $M \to X$ is an A-mapping, so also is $T(M) \to X$. A similar statement holds for $U(M)$ in case X is commutative.

Additional algebras over A can be constructed by taking a submodule of $T(M)$ or $U(M)$ forming the A-ideal it generates, and passing to the quotient algebra. It is easily seen that any A-algebra can be obtained as such a quotient.

In the special case where A is the algebra $\mathscr{A}_p$ of reduced powers, only certain M's are admissible, namely, those which satisfy the dimensionality restriction 4.9: $\mathscr{P}^i x = 0$ whenever $2i > \dim x$. Moreover, in forming $U(M)$, we must increase the ideal N so as to include all elements of the form

$$(9.3) \quad \mathscr{P}^k x - (x \otimes x \otimes \ldots \otimes x)\ (p \text{ factors}) , \quad x \in M_{2k} .$$

This insures that the relation 4.8, namely, $\mathscr{P}^k y = y^p$ is valid for $y \in U(M)_{2k}$. (It is a pleasant exercise in the use of the Adem-Cartan relations to show that N is an $\mathscr{A}_p$-module.) With these modifications, the resulting $U(M)$ is meaningful for algebraic topology.

10. Reformulation of the problem.

We are now in a position to formulate a problem similar to the one posed in section 2, but having a better chance of a positive solution. Recall that the algebra $F(R, q)^\infty$ of section 2 is small in that it has a single generator but is otherwise as big as possible subject to being commutative and associative. We found that, for many q's, it is not an $\mathscr{A}_p$-algebra, and hence cannot be realized. In analogy, we shall construct $U(Z_p, q)$ the free, commutative, associative $\mathscr{A}_p$-algebra on one generator of dimension q.

In $\mathscr{A}_p$, let $N(q)$ be the left ideal spanned by monomials in β and the $\mathscr{P}^i$ each of which has a factored form $Q' \beta^\varepsilon \mathscr{P}^k Q$ where $2k + \varepsilon > q + \deg Q$ and $\varepsilon = 0$ or 1. By 4.9, any such a monomial gives zero when applied to a q-dimensional class. Set $M(q) = \mathscr{A}_p/N(q)$ and define dimension by adding q to the degree in $\mathscr{A}_p$. Then $M(q)$ is an $\mathscr{A}_p$-module, the admissibility condition 4.9 holds, it has a single $\mathscr{A}_p$-basis element of dimension q, and it is the largest admissible $\mathscr{A}_p$-module on one element of dimension q. Finally, set $U(Z_p, q) = U(M(q))$ as defined in section 9.

If now we ask whether $U(Z_p, q)$ is realizable, the answer is Yes! It has been proved by Cartan [7] that $U(Z_p, q)$ is isomorphic as an $\mathscr{A}_p$-algebra to the cohomology algebra of the Eilenberg-MacLane complex $K(Z_p, q)$.

Having succeeded in realizing the free $\mathscr{A}_p$-algebra on one generator, it is natural to ask if quotients of this algebra can be realized. For example, choose a $y \in U(Z_p, q)$ and let W be the quotient by the minimal $\mathscr{A}_p$-ideal containing y. As an approach to this question, let D be the canonical bundle over $K(Z_p, q)$ with y as its k-invariant. Precisely, the element $y \in H^r(K(Z_p, q), Z_p)$ determines a mapping $f\colon K(Z_p, q) \to K(Z_p, r)$ such that y is the image of the fundamental class of $K(Z_p, r)$. Let E be the acyclic fibre space over $K(Z_p, r)$ with fibre $K(Z_p, r - 1)$. Then D is defined to be the fibre space over $K(Z_p, q)$ induced by E and f.

Unfortunately the complete structure of $H^*(D; Z_p)$ is not known. It is obvious that the projection $g\colon X \to K(Z_p, q)$ satisfies $g^* y = 0$. Therefore the kernel of g^* contains the $\mathscr{A}_p$-ideal generated by y. It is a reasonable conjecture that they coincide, and that the $\mathscr{A}_p$-algebra W on one generator and one relation is contained in $H^*(D; Z_p)$. It is definitely known that W is not all of $H^*(D; Z_p)$. To see this, it is only necessary to recall that the elements of $H^*(K(Z_p, q); Z_p)$ can be interpreted as primary cohomology operations, and the elements of $H^*(D; Z_p)$ as secondary operations defined on cohomology classes annihilated by y (see [1]). Numerous non-trivial secondary operations have been found.

Thus to realize W as the cohomology algebra of a space, we must modify D so as to eliminate the unwanted elements of $H^*(D; Z_p)$. But before trying this, we should reexamine our objective. It was to construct a space whose cohomology has a single generator and is maximal subject to a single relation. In one sense D already satisfies our requirement. If we admit *secondary* cohomology operations as well as the primary operations $\mathscr{A}_p$, then the g^*-image of the generator of $H^*(K(Z_p, q); Z_p)$ does in fact generate $H^*(D; Z_p)$, and the latter is free in the sense that there are no accidental relations. This is a restatement of the identification of elements of $H^*(X; Z_p)$ with secondary operations associated with y.

Thus, in attempting to realize W, we have tacitly assumed that we know what is meant by " one generator subject to one relation ". Our prejudices have again interposed themselves. The correct procedure is to analyse fully the structure of $H^*(D; Z_p)$, and then we may know how to define the concept of one generator subject to one relation.

Eventually we will want to know how to describe algebraically the cohomology algebra on k generators subject to r_1 primary relations, r_2 secondary relations, etc. We know already how to realize this algebra using Eilenberg-MacLane complexes and the fibre space constructions of Postnikov [16]. But we are a long way from being able to describe the algebra in direct algebraic terms.

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