CONVEXITY ON GRASSMANN MANIFOLDS[1])

by Herbert Busemann

The definition of a convex function presupposes that the range of the independent variable is a convex subset of a linear space. However, the question of what convexity means on ranges of different types, in particular on Grassmann manifolds, arises naturally, for example in the search for an analogue to the Legendre Condition for a-dimensional variational problems in an n-dimensional space when $1 < a < n - 1$. Thus the first new concept of convexity—total convexity in our language— occurs implicitly in the work of Carathéodory on such problems. A general and detailed discussion of convexity from the point of view of the calculus of variation was given by Wagner [11].

After the concept of convexity on Grassmann manifolds has been analyzed, it appears that *the theory of convex bodies is ready for a significant extension*, in that many theorems on convex bodies are probably special cases of more general theorems dealing with these new concepts of convexity.

Before entering a detailed discussion a *typical problem* concerning the meaning of convexity will be useful. Consider a convex body K with interior points in the n-dimensional euclidean space E^n. For a given simple a-vector ($1 \leq a \leq n - 1$) $\mathfrak{A} \neq 0$, let $P_1(\mathfrak{A})$ be the a-dimensional area of the projection of K on an a-flat parallel to $\mathfrak{A}$ and define

$$P(\mathfrak{A}) = |\mathfrak{A}| P_1(\mathfrak{A}), \quad P(O) = 0,$$

where $|\mathfrak{A}|$ is the absolute value (also called norm or area) of $\mathfrak{A}$. Clearly

$$P(k\mathfrak{A}) = |k| P(\mathfrak{A}) \quad \text{for all real } k.$$

For $a = 1, n - 1$ the a-vectors form linear spaces; in both cases it was proved by Minkowski that $P(\mathfrak{A})$ is a convex function (compare [5, pp. 45, 51]).

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[1]) Talk delivered at the Zurich Colloquium on Differential Geometry and Topology, June 1960.

For $1 < a < n - 1$ it has *a priori* no meaning to say that $P(\mathfrak{A})$ is convex, because the simple a-vectors form a Grassmann cone and not a linear manifold. We wish to find out whether we can *attribute a reasonable meaning to this statement*.

We will find that *there are three, possibly four, significant, increasingly stronger concepts of convexity*, the weakest is convexity in any linear subspace lying entirely on the Grassmann cone and follows for $P(\mathfrak{A})$ at once from Minkowski's result, the strongest is the total convexity found by Carathéodory.

First some *notations*. We use the same symbol for a vector space and the corresponding affine space. V^n is the n-dimensional affine space or vector space over the reals, V_a^n the space of a-vectors $1 \leq a \leq n - 1$, Z_a the origin of V_a^n and G_a^n the Grassmann cone in V_a^n consisting of all simple a-vectors, so that $V^n = V_1^n = G_1^n$ and $V_{n-1}^n = G_{n-1}^n$. We will encounter explicitly only simple vectors so that symbols like $\mathfrak{A}_j$ and $\sum_{j=1}^{k} \mathfrak{A}_j$ *always mean simple vectors*.

Let V^n be realized as the set of n-tuples $x^1, ..., x^n$ of real numbers. A *euclidean metrization* of V^n has the form

$$\left[\Sigma g_{ik} (x^i - y^i)(x^k - y^k) \right]^{\frac{1}{2}} ,$$

where $\Sigma g_{ik} x^i x^k$ is, of course, positive definite. $\left[\Sigma (x^i - y^i)^2 \right]^{\frac{1}{2}}$ is the *trivial* euclidean metric. Each euclidean metrization of V^n induces a euclidean metric in V_a^n, the trivial metric in V_a^n is induced by that in V^n.

A hyperplane in V_a^n considered as affine space is *simple* or a *Grassmann plane*, if the coefficients of its equation (except for the constant term) satisfy the Plücker identities or form a simple a-covector. In terms of the trivial metric this means that the *perpendicular through Z_a to the hyperplane falls on* G_a^n. A function defined in a subset of V_a^n is continuous if it is continuous with respect to one, and then all, euclidean metrics induced in V_a^n.

The sources for the following discussion are the already mentioned paper [11] of Wagner, [6], [7] and principally [8].

Consider a continuous function $f(\mathfrak{A})$ defined on G_a^n. For simplicity, and because this is the most important case, we

assume that $f(\mathfrak{A})$ is positive and positive homogeneous of degree 1, i.e.,

$$f(\mathfrak{A}) > 0 \ \textit{for} \ \mathfrak{A} \neq 0 \ \textit{and} \ f(k\mathfrak{A}) = kf(\mathfrak{A}) \ \textit{for} \ k \geqq 0 \ . \quad (1)$$

The weakest form of convexity was already mentioned: $f(\mathfrak{A})$ is *convex* if it is convex in each linear subspace lying entirely in G_a^n. This is equivalent to

$$f(\mathfrak{A}_1 + \mathfrak{A}_2) \leqq f(\mathfrak{A}_1) + f(\mathfrak{A}_2) \ , \quad (2)$$

because the simplicity of $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_1 + \mathfrak{A}_2$ implies that the a-flats parallel to $\mathfrak{A}_i$ through Z_a intersect in an $(a-1)$-flat. For $a = 1, n-1$ the relation (2) yields

$$f(\sum_{j=1}^{k} \mathfrak{A}_j) \leqq \sum_{j=1}^{k} f(\mathfrak{A}_j) \quad \text{for all} \quad k \geqq 2 \ , \quad (3)$$

but (2) does *not* imply (3) for $1 < a < n-1$. *If* f $(\mathfrak{A})$ *satisfies* (3) *we call it extendably convex*, because (3) is clearly necessary and sufficient for $f(\mathfrak{A})$ to be extendable to a convex function defined in all of V_a^n.

This condition may be rephrased. In V_a^n we consider the indicatrix I_f of $f(\mathfrak{A})$ which is the locus on G_a^n defined by $f(\mathfrak{A}) = 1$. The function $f(\mathfrak{A})$ is extendably convex if and only if I_f lies on the boundary of its own convex closure in V_a^n, hence if and only if I_f possesses at each point a supporting hyperplane in V_a^n.

Such a hyperplane will in general not be simple. *If* I_f *possesses at each point a simple supporting hyperplane, we call* f $(\mathfrak{A})$ *totally convex*.

These concepts would be of very little interest if they did not have *interesting geometric meanings in* V^n. It is wellknown that Minkowski interpreted $f(\mathfrak{A})$ for $a = 1$ as a length, less known that he also studied $f(\mathfrak{A})$ for $a = n-1$ as an area (see [5, p. 64]). We extend his ideas to general a.

Let $\rho_a(M)$ be the a-dimensional Lebesgue measure induced by a euclidean metrization of V^n of a set M in an a-flat. For a given a-vector $\mathfrak{A} \neq 0$ we consider the oriented a-flats A parallel to $\mathfrak{A}$ and in each A the open sets, or closures of open sets, M oriented as A and satisfying $\rho_a(M) = |\mathfrak{A}|$[1]). This condition

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1) For $0 < \rho_a(M) < \infty$ it is always assumed that M is open or the closure of an open set.

is independent of the choice of the euclidean metric. The totality of all sets M obtainable from $\mathfrak{A}$ in this way is denoted by $[M]$ and may be identified with $\mathfrak{A}$ because $\mathfrak{A}$ and $[M]$ determine each other uniquely. $[M] = \mathfrak{A}$ entails $[-M] = -\mathfrak{A}$. With the vanishing a-vector we identify the totality of all Borel sets M with $\rho_a(M) = 0$ and lying in a-flats.

A function $f(\mathfrak{A})$ (satisfying (1)) defines an affine a-area through

$$\alpha(M) = f(\mathfrak{A}) \quad \text{if} \quad [M] = \mathfrak{A} \;.$$

In general $\alpha(M) \neq \alpha(-M)$ because we do not assume $f(\mathfrak{A}) = f(-\mathfrak{A})$. If $f(\mathfrak{A}) \equiv f(-\mathfrak{A})$ we call f and the area α *symmetric.*

Consider an oriented closed a-dimensional polyhedron P in V^n with oriented a-faces $F_0, ..., F_k$. *If*

$$\alpha(-F_0) \leqq \sum_{1=j}^{k} \alpha(F_j) \qquad (m)$$

for any P, we say that *the* a-*flats minimize* f-*area or* α-*area.* If $[F_j] = \mathfrak{A}_j$ then

$$\sum_{j=0}^{k} \mathfrak{A}_j = 0 \quad \text{or} \quad -\mathfrak{A}_0 = \sum_{j=1}^{k} \mathfrak{A}_j$$

so that (m) *is* equivalent to

$$f\left(\sum_{j=1}^{k} \mathfrak{A}_j \right) \leqq \sum_{j=1}^{k} f(\mathfrak{A}_j)$$

and is therefore *implied by extendable convexity. The converse is not known for* $1 < a < n - 1$ and amounts to an involved semitopological question. The difficulties may be seen from the fact that for $a = 2$, $n = 4$ any 7 simple 2—vectors are dependent, but any two planes parallel to two of these 2-vectors will in general intersect in points only. This means that comparatively few sets of 2-vectors whose sum vanishes correspond to closed polyhedra, so that (m) provides rather little information on general sums $\Sigma \mathfrak{A}_j$.

In the cases $a = 1$, $n — 1$ extendable convexity becomes ordinary convexity and is easily seen to be equivalent to (m) [1]). The result goes back to Minkowski. Therefore, when $1 < a < n — 1$, convexity of $f(\mathfrak{A})$ is equivalent to (m) for P which lie in $(a + 1)$-flats.

For $1 < a < n — 1$ the condition (m) *is the natural analogue to the Legendre Condition.* It it should prove weaker than extendable convexity—the evidence available at present speaks against this possibility—it would therefore be the *fourth* significant form of convexity to which we alluded above. *Extendable convexity is probably the relevant concept for the theory of convex bodies.*

The interpretation of total convexity in V^n requires the concept of total normality or total transversality, which is at first difficult to grasp. In order to at least avoid the complications arising from orientation we assume that f or the area α is symmetric, i.e., that $f(\mathfrak{A}) = f(— \mathfrak{A})$ and indicate later the changes necessary for non-symmetric f.

In the simplest case $a = 1$ the line L is normal to the hyperplane H at the origin $z = L \cap H$ of V^n if for any set $M \subset L$ and any other line L' through $L \cap H$ the projection M' of M parallel to H on L' satisfies $\alpha(M') \geqq \alpha(M)$. (Figure 1.) If we choose as M a segment of α-length 1 beginning at z, its endpoint q lies on the indicatrix I_f of f, and we see that L is normal to H if and only if the hyperplane H_q parallel to H through q is a supporting hyperplane of I_f, because only then will the endpoint q of the projection of M lie on or outside of I_f for all choices of L'. Normality of L to H is the term used in Banach space theory, whereas in the calculus of variations one speaks of transversality of H to L. *Thus a length induces a concept of normality and transversality. We generalize this to arbitrary* a, *leaving as much free play for the dimensions as possible.*

If an a-dimensional affine area f or α is given, $1 \leqq a \leqq n — 1$, we wish to define when the a-*flat* A *is totally normal to the* b-*flat* B, $1 \leqq b \leqq n — 1$ *at the* d-*flat* D $= A \cap B$, d $<$ min (a, b), in

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[1]) The conditions for the equality sign are different for $a = 1$ and $a = n — 1$, see [8, p. 49].
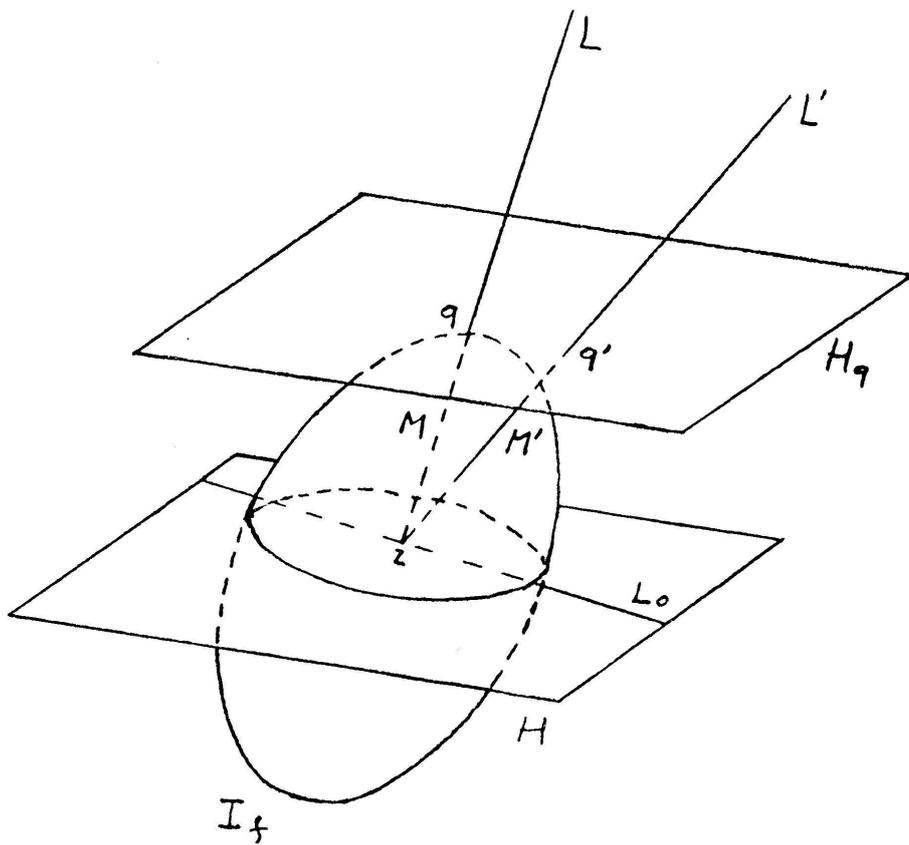
Fig. 1.

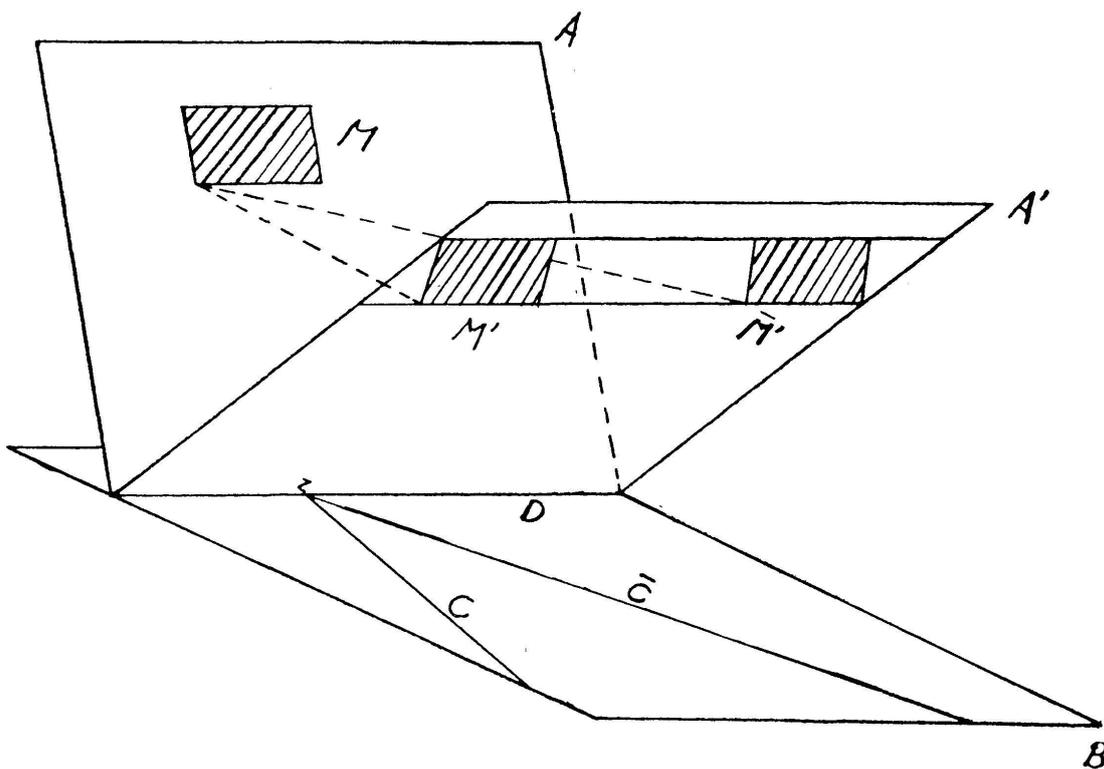Fig. 2.

$A + B = Q$, $q = \dim Q = a + b - d \leqq n$. We write this relation as

$$A \frac{D,Q}{t} \Big| B \quad \text{or} \quad B \Big| \frac{D,Q}{t} A , \qquad (4)$$

but read the second relation as: B *is totally transversal to* A *at* D *in* Q.

We distinguish two cases:

1) $d = 0$. Then $A \cap B$ is a point, it is convenient to always take the same point, say the origin z. Let M be a set in A with $0 < \alpha(M) < \infty$, and consider any a-flat A' in Q with $A' \cap B = z$ (then $A' + B = Q$) and the projection M' of M on A' parallel to B. *The relation* (4) *means that*

$$\alpha(M') \geqq \alpha(M) \qquad (5)$$

for any choice of A'.

This inequality is independent of the choice of M.

2) $d > 0$. (The case $d = 1$, $a = b = 2$ is illustrated in Figure 2.) Assuming $D \ni z$ we choose in B a $(b - d)$-flat C with $C \cap D = z$. Then $C + A = Q$, $C \cap A = z$ and we define (4) by $A \frac{z,Q}{t} \Big| C$. This definition seems to depend on the choice of C, but does not. If C is replaced by another $(b - d)$-flat $\overline{C} \subset B$ with $\overline{C} \cap D = z$, then the projections M' and $\overline{M}'$ of $M \subset A$ on A' using C and $\overline{C}$ respectively, have the same measure: $\alpha(M') = \alpha(\overline{M}')$, because it is easily seen that $\rho_a(M') = \rho_a(\overline{M}')$. Therefore (5) for $\overline{M}'$ entails (5) for M' and conversely.

When A and B are known then D and Q are, and we omit them in (4). If $a + b = n$ then $Q = V^n$ is known and will be omitted etc.

The existence of A totally normal to B for given B, D, Q follows at once from the assumption that $f(\mathfrak{A})$ is positive and continuous. We saw already in the simplest case $d = 0$, $a = 1$, $b = n - 1$, $q = n$ that the existence of B totally transversal

to A for given A, D, Q is a convexity property.   In the general case we find:

f ($\mathfrak{A}$) *or* α *is totally convex if and only if a* B *totally transversal to* A *for given* D $\subset$ A $\subset$ C *exists for all dimensions* a, b, d, q *satisfying* d $<$ min (a, b), q $=$ a $+$ b $-$ d $\leqq$ n.

There are two simple, but very important facts, relating normality for different dimensions, the second of which was used already in a special case when reducing $d > 0$ to $d = 0$.

(6)   *If* A *is (totally) normal to* B *at* D *and* d $<$ b$'$ $<$ b *then* A *is (totally) normal to any* b$'$-*flat* B$'$ *through* D *in* B.   ($L -| L_0$ in Figure 1.)

(7)   *If* A *is (totally) normal to* B *at* D *and the* b$'$-*flat* B$'$ *lies in* A $+$ B *and contains* B *but not* A, *then* A *is (totally) normal to* B$'$ *at* D$'$ $=$ B$'$ $\cap$ A.

The parentheses about " totally " indicate that (6) and (7) are also valid for the ordinary normality to be defined presently.

Repeated application of (6) and (7) shows that $f(\mathfrak{A})$ *is totally convex if for a given* a-*flat* A *an* (n $-$ a)-*flat transversal to* A *at a point exists.*   This special case is Carathéodory's form of the Legendre Condition [9, paper XX], which is therefore *equivalent to total convexity.*

It is clear—and this is Carathéodory's starting point [9, p. 364] —that total convexity implies (m).   With the previous notations, if A is the a-flat containing the face F_0 of P and B is an $(n - a)$-flat totally transversal to A at a point, then by the definition of total transversality the projection F'_j of F_j ($j \geqq 1$) parallel to B on A satisfies $\alpha(F'_j) \leqq \alpha(F_j)$.   For topological reasons

$$\bigcup_{j=1}^{k} F'_j \supset F_0, \text{ hence } (m).$$

If A is totally normal to B at D then A is by (6) totally normal to every $(d + 1)$-flat in B through D.   The converse holds trivially if $b = d + 1$, less trivially, when $a = d + 1$; and $d = $ min $(a, b) - 1$ is the *only* case where the converse is always true.   This leads us to the definition of normality.

A *is normal to* B *at* D *in* Q *or* B *is transversal to* A *at* D *in* Q, *in formulae*

$$A \overset{D,Q}{\underset{\rule{2cm}{0.4pt}}{\big|}} B \quad or \quad B \overset{D,Q}{\underset{\rule{2cm}{0.4pt}}{\big|}} A \,,$$

if A *is totally normal to every* (d + 1)-*flat through* D *in* B.

Thus total normality and normality are in general equivalent only for $d = \min(a, b) - 1$. As stated before, (6) and (7) hold for normality, so that we can draw the analogous conclusion, namely that B transversal to A for given A, D, Q exists when an $(n - a)$-flat transversal to a given a-flat at a point exists. This special case is Wagner's concept of transversality.

B *transversal to* A *for given* A, D, Q *exists if and only if* f ($\mathfrak{A}$) *is convex. It suffices for convexity that* B *exists in the case* d = min) a, b) — 1 *or even only for* b = 1 (*hence* $d = 0$, $q = a + 1$).

The relations between the various concepts are summarized in the Table.

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TABLE

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Convexity	Extendable Convexity	Total Convexity
Def:	Def:	
$f(\mathfrak{A}_1 + \mathfrak{A}_2) \leqq f(\mathfrak{A}_1) + f(\mathfrak{A}_2)$	$f\left(\sum_{j=1}^{k} \mathfrak{A}_j\right) \leqq \sum_{j=1}^{k} f(\mathfrak{A}_j)$	
$\downarrow\ \uparrow$	$\downarrow\ \uparrow$	
$f(\mathfrak{A})$ convex in each $(a+1)$-flat	$f(\mathfrak{A})$ extendable to a convex function in V_a^n	
$\downarrow\ \uparrow$	$\downarrow\ \uparrow$	
	I_f possesses supporting hyperplanes in V_a^n	Def: I_f possesses simple support- ing hyperplanes, in V_a^n
	$\downarrow\ \ \uparrow$?	$\downarrow\ \uparrow$
(m) for P in $(a+1)$-flats	(m) for general P	
$\downarrow\ \uparrow$		
Existence of B with		Existence of B with
$B \overset{D,Q}{\underset{\rule{1.5cm}{0.4pt}}{\big|}} A$		$B \overset{D,Q}{\underset{t}{\big|}} A$
for given A, D, Q		for given A, D,
$\downarrow\ \uparrow$		
Existence of B with		
$B \overset{D,Q}{\underset{t}{\big|}} A$		
if $d = \min(a, b) - 1$		

If f is not symmetric ($f\,(\mathfrak{A}) \not\equiv f\,(-\,\mathfrak{A})$) then in the definition of (total) normality the flat B remains non-oriented, A is oriented and A' has the orientation induced by that of A through projection parallel to B. The values $\alpha\,(M)$ and $\alpha\,(M')$ are determined by these orientations.

Our exposition seems to justify the restriction to the case $b = n - a$, $d = 0$ considered by Carathéodory and Wagner, but in discussing symmetry and equivalence of normality we will find that the other cases prove most interesting. The answer to the equivalence problem requires the concept of quadratic area: $f\,(\mathfrak{A})$ *or* α *is quadratic if* $f^2\,(\mathfrak{A})$ *is a quadratic form in the components of* $\mathfrak{A}$. Clearly a quadratic area satisfies $f\,(\mathfrak{A}) = f\,(-\,\mathfrak{A})$ and is extendably convex, but need *not* be totally convex. A euclidean area, i.e. an area ρ_a induced by some euclidean metrization of V^n is quadratic, the converse holds only for $a = 1$ and $a = n - 1$. Therefore *a quadratic area is euclidean in every* (a $+$ 1)-*flat but not necessarily in every* (a $+$ 2)-*flat*. In fact, if the latter is the case, then the area is euclidean.

If a convex or totally convex area α is given then the relations

$$A \left.\frac{D}{}\right| A' \quad \text{or} \quad A \left.\frac{D}{t}\right| A'$$

are defined for two a-flats A, A' with $D = A \cap A'$ provided $d < a$ and $2a - d \leqq n$. It is natural to ask when these relations are symmetric. Since for $f\,(\mathfrak{A}) \not\equiv f\,(-\,\mathfrak{A})$ the flat A has to be taken as oriented and A' is non-oriented the problem is natural only for symmetric areas. Our preceding remarks show that for $d = a - 1$ the normality relation is symmetric when the area is quadratic.

The problem is wellknown for $a = 1$ and was solved long ago (1916) by Radon and Blaschke. A convex 1-area is a Minkowski metric and our space is a finite-dimensional Banach space. The answer to the problem is as follows: if $n = 2$, then symmetry of normality between lines does not imply that the metric is euclidean (Radon [10]), but for $n > 2$ it does (Blaschke [3]). These results were rediscovered by the Banach space people in the 1940's and formulated in their own language.

In our case we have more free play and the problem presents itself in the following form:

Let a symmetric (totally) convex a-area α and a symmetric (totally) convex b-area β be given, moreover fixed numbers d, q *such that* $d < a \leqq b < q = a + b - d \leqq n$. *Then* α *defines the relation* $A \left.\dfrac{D}{(t)}\right| B$. *When is this relation equivalent to* $B \left.\dfrac{D}{(t)}\right| A$ *which is defined by* β ?

The answer is this:

If the two relations are equivalent, then both α and β are quadratic, unless $a + b = n$ *and* $d = 0$.

In the latter case, when a totally convex a-area is given, an up to a constant factor unique totally convex (n — a)-*area can be constructed such that the relations* $A \left.\dfrac{z}{t}\right| B$ *and* $B \left.\dfrac{z}{t}\right| A$ *are equivalent.*

The special case $\alpha \equiv \beta$ leads to symmetry of normality and we find:

If α is a (totally) convex a-area and if $A \left.\dfrac{D}{(t)}\right| A'$ *implies* $A' \left.\dfrac{D}{(t)}\right| A$ *then α is quadratic unless* n = 2a *and* d = 0. *In the latter case there are for each* $a \geqq 1$ *non-quadratic totally convex areas for which* $A \left.\dfrac{z}{t}\right| A'$ *implies* $A' \left.\dfrac{z}{t}\right| A$.

For $a = 1$ *all* areas or Minkowski metrics with this property are known through Radon [10]; in such generality the problem has not been solved for $a > 1$.

In special cases we can assert that a (totally) convex area α is euclidean if normality of a-flats at d-flats is symmetric, namely when $2a < n$ and $d = 0$ or $2a > n$ and $d = 2a - n$.

Not all results for special dimensions have interesting extensions. An example is provided by the *integral geometry of (symmetric) affine areas* (for the following see [7]). Since affine areas are invariant only under translations, one would conjecture that a true integral geometry exists only for the translation group. Such a theory does exist, but is not essentially new,

because there is a simple method of transcribing the results of
the theory for the euclidean areas and this group, as it is deve-
loped in Berwald and Varga [2], directly without new proofs so
that they become valid for affine areas. The only surprise is,
that *results into which areas of different dimensions enter remain
valid for completely unrelated, and not even necessarily convex
affine areas.*

Contrary to our conjecture Blaschke showed in [3] that *the
Minkowski plane possesses a much richer integral geometry*, in
which most of the euclidean results for the full group of motions
of E^2 hold. The central point is the *existence of a positive
density dL* for all lines in the plane, such that for any segment S
the relation

$$\alpha(S) = \int_{L \cap S \neq 0} dL \; ^1)$$

holds. Thus one might hope (and Blaschke predicted in [3])
that there is a rich integral geometry for a reasonably general
a-dimensional area α. As criterion we take the existence of a
positive density dB for the $(n - a)$-flats such that for any
convex set S in an a-flat the relation

$$\alpha(S) = \int_{B \cap S \neq 0} dB$$

holds. *If* n > 2 *such a density exists only if α is extendably
convex* (which is a reasonable condition) *and has in addition a
special form* which deprives the theory of much of its interest.

We come to the *new problems* arising *in the theory of convex
bodies* through the new concepts. The classical theory proves
for many functions derived from one or several convex bodies
in E^n and depending on the position of a hyperplane, that they
are convex. As our example of the projection function $P(\mathfrak{A})$
shows these functions are often naturally defined for a varying
a-flat, and the classical results only contain the convexity of the
new functions, and not extendable or total convexity.

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1) The normalization of dL or dB differs from the usual one by a constant factor.
We emphasize that dL and dB be *positive because this condition causes the trouble for dB
when $n > 2$.*

No method for establishing total convexity is known.[1] 7] shows *that in various cases extendable convexity can be proved with the help of integral geometry.* The following is a typical example: Let F be a closed convex hypersurface in E^n. For a given simple a-vector $\mathfrak{A} \neq 0$, $(a \geqq 2)$, take the family of oriented a-flats L_A parallel to $\mathfrak{A}$, let dL_A be their density and form, with ρ_{a-1} meaning ordinary $(a-1)$-dimensional area,

$$J(\mathfrak{A}) = |\mathfrak{A}| \cdot \int_{L_A \neq F \cap 0} \rho_{a-1}(L_A \cap F)\, dL_A, \quad J(O) = 0.$$

Then $J(k\mathfrak{A}) = |k| J(\mathfrak{A})$ and $J(\mathfrak{A})$ is extendably convex because

$$\int_{L_A \neq F \cap 0} \rho_{a-1}(L_A \cap F)\, dL_A = \int_F |\mathfrak{n}_p \wedge \mathfrak{A}^s|\, dF_p,$$

where $\mathfrak{n}_p$ is the exterior unit normal to F at p, dF_p is the area element of F and $\mathfrak{A}^s$ is the supplementary $(n-a)$-vector to $\mathfrak{A}$. Extendable convexity follows from $|\sum \mathfrak{n}_p \wedge \mathfrak{A}_j^s| \leqq \sum |\mathfrak{n}_p \wedge \mathfrak{A}_j^s|$.

W. Fenchel and the author noticed that the examples in [7] can be subordinated to a general principle, which is not discussed here because it is not yet clear how far it carries.

Whether the projection function $P(\mathfrak{A})$ is extendably convex is not known. Those familiar with the theory of convex bodies will recognize this as a *novel problem*, because

$$P(\mathfrak{A}) = \tbinom{n}{a} V_{n-a}(K, M), \ if \ M, \ convex \ and \ [M] = \mathfrak{A}^s.$$

As usual (see [5, p. 40] $V_a(K, M)$ is the mixed volume of K taken $n-a$ times and M taken a times. The just mentioned general principle proves $V_{n-1}(K, M) = Q(\mathfrak{A})$ to be extendably convex.

We conclude with an interesting and probably quite dificult convexity problem. In its simplest form it is this: Let K be a convex body in E^n with z as interior point and center. For an a-vector $\mathfrak{A} \neq 0$ let A be the a-flat parallel to $\mathfrak{A}$ through z and put

$$\alpha(\mathfrak{A}) = |\mathfrak{A}|/\rho_a(A \cap K), \quad \alpha(O) = 0.$$

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[1] This difficulty was already encountered by Carathéodory when he tried to establish his transversality condition for given variational problems.

This function is convex; the proof of convexity for the problem in a slightly more general form implies the Brunn-Minkowski Theorem (the relation between the two theorems is discussed in Barthel [1]). *This shows that the extendable convexity of* α ($\mathfrak{A}$), *if true, is a deep theorem with many implications.*

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