To verify this we need slight extensions of Lemma 2, and of the fact that derivatives possess the Darboux property.

LEMMA 2′. *If* f *is continuous and* f′ *exists (finite or infinite) in* (p, q), *while* $f(p^+)$ *does not exist (finite or infinite); if* G *is continuous in* [p, q), G′ *exists (finite) in* (p, q) *and* $G'(x) \neq 0$ *in* (p, q)*; then* f′ (x)/G′ (x) *assumes every finite value in* (p, q).

Since $f(p^+)$ does not exist, $H(x) = f(x) - \lambda G(x)$ does not approach a limit (since $G(p^+)$ does exist). Hence H is not monotonic and so possesses extrema. At an extremum ξ we have $H'(\xi) = 0$, so $f'(\xi) = \lambda G'(\xi)$. Since $G'(\xi)$ is neither 0 nor infinite, $f'(\xi)/G'(\xi) = \lambda$.

LEMMA 3. *If* f *and* G *are continuous in* [p, q]*; if* f′ *exists (finite or infinite) in* [p, q], *and* G′ *exists (finite) in* [p, q]*; if* f′ (p) *and* f′ (q) *are finite and* G′ *has a fixed sign (and hence is never* 0*) in* [p, q]*; and if*

$$f'(p)/G'(p) < c < f'(q)/G'(q),$$

then there is a ξ *in* (p, q) *such that* $f'(\xi)/G'(\xi) = c$.

This says in effect that f'/G', like f', has the Darboux property.

Consider $H(x) = f(x) - cG(x)$ and suppose for definiteness that $G'(p) > 0$. Then $H'(p) < 0$, $H'(q) > 0$, so the continuous function H cannot assume its minimum at p or q. If H assumes its minimum at ξ, we have $f'(\xi) = cG'(\xi)$ and so (since $G'(\xi)$ is neither zero nor infinite), $f'(\xi)/G'(\xi) = c$.

It now follows just as before that Theorems 1 and 2 hold, with $g = 1/G'$ in (2).

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