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ON THE SOLUTION OF SIMULTANEOUS IMPLICIT EQUATIONS

by Smbat ABIAN and Arthur B. BROWN, Flushing, N.Y.

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In this self-contained paper, generalizing the results obtained in an earlier paper [1] on the case of a single implicit equation, the authors give an explicit method for solving a system of p simultaneous implicit equations $f_i(x_1, \dots, x_n, y_1, \dots, y_p) = 0$ for the p unknown functions $y_i = Y_i(x)$. The method consists of successive substitutions.

The hypotheses of the classical implicit function theorem are replaced by weaker hypotheses. In particular, the functions f_i are not required to be differentiable, and there is no requirement that a known point satisfy the given equations.

Two appraisals of the remainder error at the m th stage of approximation are given, one of which is valid regardless of errors made at earlier stages of the computation. It is also proved that if the given functions f_i satisfy Lipschitz conditions in a certain subset of the x 's, then the $Y_i(x)$ will also satisfy Lipschitz conditions in the same subset.

Throughout the paper, unless otherwise specified, the indices i, j, k run from 1 to p , the index r runs from 1 to n , $(x) \equiv (x_1, \dots, x_n)$ and $(y) \equiv (y_1, \dots, y_p)$. All functions and variables are understood to be real, and the functions singlevalued.

Theorem 1. Given a set of p functions $f_i(x_1, \dots, x_n, y_1, \dots, y_p) \equiv f_i(x, y)$ continuous on the closed region $N_1 \subset E^{n+p}$ determined by the relations $|x_r - a_r| \leq \alpha_{r1}$, $|y_i - b_i| \leq \beta_{i1}$, where α_{r1} , β_{i1} are positive constants, let there exist a non-singular matrix of constants (C_{ij}) and a matrix of constants (D_{ij}) with

$$\sum_j D_{ij} < 1, \quad (1)$$

such that, for $(x, y) \in N_1$,

$$|\delta_{ij} \Delta y_j + \sum_k C_{ik} \Delta_j f_k| \leq D_{ij} |\Delta y_j|, \quad (2)$$

where δ_{ij} is the Kronecker δ , Δy_j is an increment of the variable y_j and $\Delta_j f_k$ is the increment of the function f_k corresponding to the increment Δy_j of y_j .

Then there exist p positive constants $\beta_i \leq \beta_{i1}$ such that

$$\beta_i - \sum_j D_{ij} \beta_j > 0. \quad (3)$$

If furthermore $f_i(a, b) = f_i(a_1, \dots, a_n, b_1, \dots, b_p)$ satisfy

$$\left| \sum_k C_{ik} f_k(a, b) \right| < \beta_i - \sum_j D_{ij} \beta_j, \quad (4)$$

then there exist n positive constants $\alpha_r \leq \alpha_{r1}$ and a set of p continuous functions $Y_i(x)$ such that if T is the closed region of x -space determined by $|x_r - a_r| \leq \alpha_r$, the locus of the system of equations $y_i = Y_i(x)$ for $x \in T$ is the same as that of the system $f_i(x, y) = 0$ for $(x, y) \in N$, where $N \subset N_1$ is the closed region determined by

$$|x_r - a_r| \leq \alpha_r, \quad |y_i - b_i| \leq \beta_i.$$

We shall prove Theorem 1 simultaneously with Theorem 2.

Theorem 2. The constants α_r of Theorem 1 can be chosen subject only to the conditions

$$\left| \sum_k C_{ik} f_k(x, b) \right| \leq \beta_i - \sum_j D_{ij} \beta_j, \quad |x_r - a_r| \leq \alpha_r. \quad (5)$$

Furthermore if we introduce

$$F_i(x, y) \equiv y_i + \sum_k C_{ik} f_k(x, y), \quad (x, y) \in N_1, \quad (6)$$

and take $Y_i(x; 0)$ as a function, not necessarily continuous, satisfying

$$|Y_i(x; 0) - b_i| \leq \beta_i, \quad x \in T, \quad (7)$$

then for $m \geq 0$ the function

$$Y_i(x; m+1) = F_i[x, Y(x; m)], \quad (8)$$

is well defined for $x \in T$ and

$$Y_i(x) = \lim_{m \rightarrow \infty} Y_i(x; m). \quad (9)$$

Proof of Theorems 1 and 2. Before beginning the actual proof, we observe that a natural choice for $Y_i(x; 0)$ is $Y_i(x; 0) = b_i$. (Cf. Theorem 4.) We observe also that condition (2) is readily satisfied if $f_i(x, y)$ is of class C^1 and the Jacobian of the partial derivatives of the $f_i(x, y)$ with respect to the y_j is not zero at (a, b) . For in that case the matrix equation

$$(\delta_{ij}) + (C_{ik}) \left(\frac{\partial f_k}{\partial y_j} \right) = 0, \quad (x, y) = (a, b), \quad (10)$$

is solvable for (C_{ik}) , and it follows that if every D_{ij} is a positive constant, (2) will hold if N_1 is taken as a sufficiently small neighborhood of (a, b) . From (10) we infer that (C_{ik}) , so obtained, is non-singular.

Returning now to the actual proof, we first observe that, in view of (1), relations (3) are easily satisfied, for example by taking $\beta_i = \min_j (\beta_{j1})$. We now assume that the β_i have been so chosen and that (4) is satisfied.

Since the f_i are continuous, we see from (4) that constants $\alpha_r \leq \alpha_{r1}$ can be chosen so that (5) is satisfied. We assume that such constants α_r have been chosen.

Let $N \subset N_1$ be defined as in the statement of Theorem 1. If (x, y) and $(x, z) \in N$, from (6) we obtain

$$\begin{aligned} F_i(x, z) - F_i(x, y) &= z_i - y_i + \sum_k C_{ik} [f_k(x, z) - f_k(x, y)] = \\ &= \sum_j \delta_{ij} \Delta y_j + \sum_j \sum_k C_{ik} \Delta_j f_k. \end{aligned}$$

Hence, in view of (2), we infer that

$$\left| F_i(x, z) - F_i(x, y) \right| \leq \sum_j D_{ij} \left| z_j - y_j \right|, \quad (11)$$

for (x, y) and (x, z) belonging to N .

We now introduce (8) and prove inductively that, for $m \geq 0$, $Y_i(x; m)$ is well defined, and

$$|Y_i(x; m) - b_i| \leq \beta_i, \quad x \in T. \quad (12)$$

From (7) we see that (12) is true for $m = 0$. Now let us assume that (12) is true for $m = s$, so that for $x \in T$ the point $[x, Y(x; s)] \in N$. This, in view of (8), implies that $Y_i(x; s+1)$ is well defined for $x \in T$. From (6) and (5) we see that

$$|F_i(x, b) - b_i| \leq \beta_i - \sum_j D_{ij} \beta_j, \quad x \in T. \quad (13)$$

From (8) we obtain

$$|Y_i(x; s+1) - b_i| \leq |F_i[x, Y(x; s)] - F_i(x, b)| + |F_i(x, b) - b_i|,$$

a relation which, in view of (11), (12) with $m = s$ and (13), implies (12) with $m = s+1$. Hence we infer that for $x \in T$ and $m \geq 0$, $Y_i(x; m)$ is well defined, and (12) holds, so that the point $[x, Y(x; m)] \in N$.

From (8) and (11), if $m \geq 1$, we have for $x \in T$

$$|Y_i(x; m+1) - Y_i(x; m)| \leq \sum_j D_{ij} |Y_j(x; m) - Y_j(x; m-1)|. \quad (14)$$

Let

$$D = \max_i \left(\sum_j D_{ij} \right). \quad (15)$$

From (1) and (2) we see that

$$0 \leq D < 1. \quad (16)$$

From (14) and (15) we infer that, for $m \geq 1$ and $x \in T$,

$$\left[\max_i |Y_i(x; m+1) - Y_i(x; m)| \right] \leq D \left[\max_j |Y_j(x; m) - Y_j(x; m-1)| \right]. \quad (17)$$

By applying (17) with $m = 1, 2, \dots, s$ and then replacing s by m , we obtain, for $m \geq 1$ and $x \in T$,

$$|Y_i(x; m+1) - Y_i(x; m)| \leq D^m \left[\max_j |Y_j(x; 1) - Y_j(x; 0)| \right]. \quad (18)$$

For $x \in T$, the bracket on the right is bounded by $2 \max_j (\beta_j)$.

Thus, in view of (16) and (18), the sequence $\{Y_i(x; m)\}_j$, $x \in T$, is uniformly convergent for each i . Hence $Y_i(x)$, as defined in (9), exists. Moreover, from (9) and (12) we conclude that, for $x \in T$, $|Y_i(x) - b_i| \leq \beta_i$, and therefore the locus $y_i = Y_i(x)$ is contained in N .

From (9) and (8), in view of the continuity of $F_i(x, y)$ on N , we see that

$$Y_i(x) \equiv F_i[x, Y(x)], \quad x \in T. \quad (19)$$

Since (C_{ik}) is non-singular, we then infer from (6) that

$$f_i[x, Y(x)] \equiv 0, \quad x \in T. \quad (20)$$

We thus see that the locus of the system of equations $y_i = Y_i(x)$ is contained in the locus of the system of equations $f_i(x, y) = 0$, for $(x, y) \in N$.

Next we prove that, for $x \in T$, $y_i = Y_i(x)$, given by (9), gives the complete locus of the system of equations $f_i(x, y) = 0$ for $(x, y) \in N$. Suppose that $f_i(\xi, \eta) = 0$ with $(\xi, \eta) \in N$. From (6) we infer that

$$\eta_i = F_i(\xi, \eta). \quad (21)$$

From (19), (21) and (11) we have

$$|\eta_i - Y_i(\xi)| \leq \sum_j D_{ij} |\eta_j - Y_j(\xi)|,$$

and from (15) we further infer that

$$\left[\max_i |\eta_i - Y_i(\xi)| \right] \leq D \left[\max_i |\eta_i - Y_i(\xi)| \right].$$

In view of (16) we now infer that $\eta_i - Y_i(\xi) = 0$, so that $\eta_i = Y_i(\xi)$. We thus conclude that $y_i = Y_i(x)$ for $x \in T$ gives the complete locus of the system of equations $f_i(x, y) = 0$ for $(x, y) \in N$.

It remains only to prove that $Y_i(x)$ is continuous. For this purpose, take $Y_i(x; 0) = b_i$, which satisfies (7) and makes $Y_i(x; 0)$ continuous. Examination of the above proof then shows that $Y_i(x; m)$ is continuous for $m \geq 0$. Since the

sequence $\{Y_i(x; m)\}$ has been proved to be uniformly convergent for each i , we infer that $\left[\lim_{m \rightarrow \infty} Y_i(x; m)\right]$ is continuous.

But we have already shown that for each $x \in T$ there is a set of uniquely determined values $Y_i(x)$ with $|Y_i(x) - b_i| \leq \beta_i$, and satisfying (20). Hence the functions $Y_i(x)$ given by (9) are continuous, and the proof is complete.

We now give two appraisals of the remainder error.

Theorem 3. For $x \in T$ and $m \geq 1$,

$$|Y_i(x; m) - Y_i(x)| \leq \frac{D^m}{1 - D} \left[\max_j |Y_j(x; 1) - Y_j(x; 0)| \right], \quad (22)$$

$$|Y_i(x; m) - Y_i(x)| \leq \frac{D}{1 - D} \left[\max_j |Y_j(x; m) - Y_j(x; m - 1)| \right]. \quad (23)$$

Moreover, relation (23) is valid regardless of errors in computation through the $Y_i(x; m - 1)$, provided merely that $|Y_i(x; m - 1) - b_i| \leq \beta_i$ and that $[Y(x; m)]$ is calculated correctly from $[Y(x; m - 1)]$.

Proof. Since $Y_i(x) - Y_i(x; m) = [Y_i(x; m + 1) - Y_i(x; m)] + [Y_i(x; m + 2) - Y_i(x; m + 1)] + \dots$, relation (22) follows from (9), (16), (18) and the formula for the sum of a geometric series.

By comparing the given relation $|Y_i(x; m - 1) - b_i| \leq \beta_i$ with (7), we see that $[Y(x; m - 1)]$ can be considered to be a new $[Y(x, 0)]$. If we apply (22) with $m = 1$ and this new $[Y(x; 0)]$, we obtain (23).

The proof given in the preceding paragraph makes clear the truth of the final assertion of Theorem 3.

We observe that this same procedure of considering $[Y(x; m - 1)]$ to be a new $[Y(x; 0)]$ shows that a finite number of errors of calculation will not prevent the sequence $\{Y_i(x, m)\}$ from converging to the function $Y_i(x)$.

Theorem 4. If $Y_i(x; 0) = b_i$, $x \in T$, then, for $x \in T$ and $m \geq 1$,

$$|Y_i(x; m) - Y_i(x)| \leq \frac{D^m}{1 - D} \left[\max_k \left(\beta_k - \sum_j D_{kj} \beta_j \right) \right]. \quad (24)$$

Proof. With $Y_i(x; 0) = b_i$, we have, by (8), for $x \in T$,

$$Y_i(x; 1) - Y_i(x; 0) = F_i(x; b) - b_i, \quad (25)$$

Relation (24) now follows from (22), (25) and (13). This completes the proof.

Theorem 5. Under the hypotheses of Theorem 1, and with the α_r 's chosen as in Theorem 2, if the $f_i(x, y)$ satisfy Lipschitz conditions in a subset of the x_r 's, the functions $Y_i(x)$ will also satisfy Lipschitz conditions in this same subset.

Proof. With $q \leq n$ and $x_t = \xi_t$ for $t > q$, suppose that, if (x, y) and $(\xi, y) \in N$,

$$|f_i(\xi, y) - f_i(x, y)| \leq \sum_{t=1}^q H_{it} |\xi_t - x_t|, \quad (26)$$

where the H_{it} 's are non-negative constants. Since

$$|F_i[\xi, Y(\xi)] - F_i[x, Y(x)]| \leq |F_i[\xi, Y(\xi)] - F_i[x, Y(\xi)]| + |F_i[x, Y(\xi)] - F_i[x, Y(x)]|,$$

we infer from (6), (26) and (11) that

$$|F_i[\xi, Y(\xi)] - F_i[x, Y(x)]| \leq \sum_k |C_{ik}| \sum_{t=1}^q H_{kt} |\xi_t - x_t| + \sum_j D_{ij} |Y_j(\xi) - Y_j(x)|. \quad (27)$$

From (27), (19) and (15), and letting $\gamma_t = \max_i \left(\sum_k |C_{ik}| H_{kt} \right)$, we obtain

$$|Y_i(\xi) - Y_i(x)| \leq \sum_{t=1}^q \gamma_t |\xi_t - x_t| + D \left[\max_j |Y_j(\xi) - Y_j(x)| \right].$$

Therefore

$$|Y_i(\xi) - Y_i(x)| \leq \sum_{t=1}^q \frac{\gamma_t}{1 - D} |\xi_t - x_t|.$$

Hence the theorem is true.

The results above are easily applied to the problem of solving p equations $g_i(y_1, \dots, y_p) = 0$ in p unknowns, considered as a special case of the system $f_i(x, y) = 0$ in which the f_i are

independent of x . In this case the functions $Y_i(x)$ become constants Y_i . The following theorem corresponds to Theorems 1 and 2.

Theorem 6. Given the functions $g_i(y_1, \dots, y_p) \equiv g_i(y)$ continuous on the closed region $N_1 \subset E^p$ determined by the relations $|y_i - b_i| \leq \beta_{i1}$, where the β_{i1} are positive constants, let there exist a non-singular matrix of constants (C_{ij}) and a matrix of constants (D_{ij}) with $\sum_j D_{ij} < 1$, and such that, for $y \in N_1$,

$$\left| \delta_{ij} \Delta y_j + \sum_k C_{ik} \Delta_j f_k \right| \leq D_{ij} |\Delta y_j|.$$

Then there exist p positive constants $\beta_i \leq \beta_{i1}$ such that $\beta_i - \sum_j D_{ij} \beta_j > 0$. If furthermore the quantities $g_k(b) = g_k(b_1, \dots, b_p)$ satisfy

$$\left| \sum_k C_{ik} g_k(b) \right| < \beta_i - \sum_j D_{ij} \beta_j,$$

then the system of simultaneous equations $g_i(y) = 0$ has a unique solution $y_i = Y_i$ in the closed region $N \subset N_1$ determined by $|y_i - b_i| \leq \beta_i$.

Moreover, if for $y \in N_1$ we define $G_i(y) = y_i + \sum_k C_{ik} g_k(y)$, and if $Y_i(0)$ is any constant satisfying $|Y_i(0) - b_i| \leq \beta_i$, then for $m \geq 0$ the constants $Y_i(m+1) = G_i[Y(m)]$ are well defined, and $Y = \lim_{m \rightarrow \infty} Y_i(m)$.

The appraisals of the remainder error given in Theorems 3 and 4 remain valid.

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