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# THE POSSIBLE TRANSFORMATIONS OF A REAL CURVE INTO A CURVE WITH REAL EQUATION AND PASSING THROUGH THE ISOTROPIC POINTS

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1. The first problem of this paper is to investigate the possibility of transforming a curve, expressible as a function with real coefficients of the variables, into a second curve expressible by an equation in cartesian coordinates with real coefficients and such that two real points on the curve become the isotropic points. It should be noted that any real transformation of the coordinate system will not affect the reality of the equation of a curve.

Let  $A$  and  $B$  be two real points on such a curve, and suppose  $f(x, y, z) = 0$  is the equation of the curve referred to a real triangle of reference  $XYZ$  where  $X, Y$  are harmonic conjugates with respect to  $A, B$ . The lines  $ZA, ZB$  are then given by  $y \pm \lambda x = 0$ , where  $\lambda$  is some real constant. It is easy to show that if the equations of  $ZA, ZB$  are to become  $\iota y \pm x = 0$ , the necessary transform is equivalent to that in which  $x$  is replaced by  $x, y$  by  $\lambda \iota y$  and  $z$  by  $\iota z$ , where the triangle of reference remains unchanged.  $X$  and real points on  $YZ$  are the only points which remain real, and the only real lines which remain real are  $YZ$  and those through  $X$ . If we then replace  $z$  by unity,  $ZA$  and  $ZB$  become isotropic lines,  $\iota y \pm x = 0$ , in a cartesian field with real rectangular axes  $ZX, ZY$ .  $A$  and  $B$  are now the isotropic points in this field.

Suppose that in the original field we have a non-degenerate curve of degree  $N$  with real coefficients and of the form

$$(x, y, z) \equiv (U_0 x^N + U_2 x^{N-2} + U_4 x^{N-4} + \dots) + (U_1 x^{N-1} + U_3 x^{N-3} + U_5 x^{N-5} + \dots) = 0 .$$

where  $U_r$  is homogeneous of degree  $r$  in  $y$  and  $z$ . After the first transform the curve has equation

$$(u_0 x^N - u_2 x^{N-2} + u_4 x^{N-4} - \dots) + (u_1 x^{N-1} - u_3 x^{N-3} + u_5 x^{N-5} - \dots) = 0 ,$$

where  $u_r$  is homogeneous of degree  $r$  in  $y$  and  $z$  and with real coefficients. Hence, if the curve is to transform finally into one through the isotropic points and whose equation has real coefficients, the equation must initially have the form when  $N = 2n$

$$U_0 x^{2n} + U_2 x^{2n-2} + \dots + U_{2n-2} x^2 + U_{2n} = 0 , \quad (1)$$

and when  $N = 2n + 1$

$$U_1 x^{2n} + U_3 x^{2n-2} + \dots + U_{2n-1} x^2 + U_{2n+1} = 0 . \quad (2)$$

In the first case all the odd polar curves with respect to  $X$  have  $x$  as a factor, with  $YZ$  as the polar line of  $X$ . In the second case the curve passes through  $X$ , which is an inflexion: the odd polar curves with respect to  $X$  have  $x$  as a factor (corresponding to the line  $YZ$ ), the remaining part of the  $(2n - 1)$ th. polar being the inflexional tangent  $U_1 = 0$ . We will refer to  $YZ$  as the conjugate line of  $X$  in each case. If the curve has a multiple point of order  $k$  at  $X$ , its equation is of the form

$$U_k x^{N-k} + U_{k+2} x^{N-k-2} + \dots + U_N = 0 ,$$

with  $N - k$  even. The  $(k + 1)$ th. polar curve of  $X$  is  $U_k x = 0$ , that is the tangents at the multiple point (of necessity inflexional) and the conjugate line.

Rewriting the equations in descending powers of  $z$ , we have

$$V_0 z^{2n} + V_1 z^{2n-1} + \dots + V_r z^{2n-r} + \dots + V_{2n} = 0 , \quad (3)$$

or

$$V_0 z^{2n+1} + V_1 z^{2n} + \dots + V_r z^{2n-r+1} + \dots + V_{2n+1} = 0 , \quad (4)$$

where  $V_r$  is homogeneous of degree  $r$  in  $x$  and  $y$ . Since only even powers of  $x$  occur it follows that  $V_r$  is of the form

$$\prod_1^{r/2} (a_s x^2 + b_s y^2) \quad \text{when } r \text{ is even,}$$

or

$$y \prod_1^{r/2} (a_s x^2 + b_s y^2) \quad \text{when } r \text{ is odd.}$$

Hence the set of points on  $XY$ , other than  $X$ , given by any  $V_r = 0$  form an involution with  $X, Y$  as double points. In particular the points of intersection of the curve with  $XY$ , given by  $V_N = 0$ , have the same property. The expression of  $y^2 - \lambda^2 x^2$  on which the transformation was based will be a factor of  $V_N$ , that is one at least of the expressions  $a_s/b_s$  is negative, and  $A, B$  are any pair of points corresponding to such a factor  $a_s x^2 + b_s y^2$ . It follows that any curve with real equation which satisfies these conditions can be transformed into a circular curve with real equation.

2. Consider the intersections of any curve of this type with the line  $py + qz = 0$ , joining  $X$  to any point  $P$  on  $YZ$ . Their joins to  $Z$  are given by an equation of the same form as  $V_N = 0$ , and hence the intersections, other than  $X$ , form an involution with  $X$  and  $P$  as the double points. It follows at once that if  $P$  is a  $2k + 1$  ple point on the curve  $XP$  is a tangent to one of the branches. We shall refer to any real point having the properties of  $X$  with respect to the curve as a pole. We have now shown that, given a pole, any real point on the curve and its mate in the involution cut on the line joining the point to the pole can be transformed into the isotropic points, giving a circular curve whose equation has real coefficients.

If  $V_r$  occurs in the equation of the curve, the equation of the  $(N - r)$ th. polar curve with respect to  $Z$  is of the same form as (3), (4): if  $V_r$  does not occur the equation of the  $(N - r)$ th. polar curve with respect to  $Z$  contains  $z$  to some power as a factor and the remaining part is of the same form as (3, 4).  $X$  is therefore a pole of the  $(N - r)$ th. polar curve of  $Z$  or of the remaining part of it, as the case may be, where  $Z$  is any point on the conjugate line of  $X$ . Hence since the first polar curve of  $Z$  passes once through each node and not, in general through a point of contact of a tangent from  $X$ , a node and such a point of contact cannot be paired together, and similarly for other singularities. The case of two nodes collinear

with  $X$  in which one of them only has  $XY$  as a tangent similarly cannot arise. The two points which are to become the isotropic points must therefore be of exactly similar type. Further any isolated singularity must be at  $X$  or on  $YZ$ .

It should be noticed that in the case of a curve of even degree containing only even powers of  $z$  a pole at  $X$  implies also poles at  $Y$  and  $Z$ .

**3.** The second problem is to discuss the possibility of a second pole not on  $YZ$ . From the preceding results it follows that the form of the equation of the curve having a pole at  $X$  is unaltered if the triangle of reference is defined by  $X$  and any two real points on  $YZ$ . If the curve has a second pole  $X_1$  not on  $YZ$ , we can therefore take it as lying on  $XY$  with  $Z$  the common point of the conjugate lines of  $X$  and  $X_1$ . The intersection of these lines cannot be at  $Y$  since in that case  $X$ ,  $Y$  and  $X_1$ ,  $Y$  would be double points of two involutions among the same points (projecting  $Y$  to infinity,  $X$  and  $X_1$  would both become the mean centre of the same points). The conjugate lines of  $X$  and  $X_1$  are therefore distinct and lead to a definite point of intersection  $Z$ .

It follows at once that the points  $X$  and  $X_1$  are poles for all polar curves with respect to  $Z$  of the initial curve and lie on all such curves of odd degree. The points of intersection of  $V_r = 0$  with  $XY$  for any value of  $r$  (but excluding  $X$  if  $r$  is odd) may therefore be paired as an involution having  $X$  as a double point (the other double point being  $Y$ ) or, alternatively excluding  $X_1$  if  $r$  is odd they may be paired as an involution having  $X_1$  as a double point. In this case the second double point is the intersection,  $Y_1$ , of the conjugate line of  $X_1$  with  $XY$ . Further, since a real point cannot be paired with an imaginary point in any involution with real double points, nor can a point of simple intersection be paired with one of multiple intersection, the various types of points must be grouped together to form sub-groups. Each sub-group must separately form involutions with  $X$ ,  $Y$  and  $X_1$ ,  $Y_1$  as double points. The expression leading to a sub-group of  $s$  points will be denoted by  $\varphi_s$  having the same form as  $V_s$  and

$$V_r \equiv \varphi_s \cdot \varphi_t \dots \quad \text{where} \quad r = s + t + \dots$$

in which some of  $s, t, \dots$  may be equal. The existence of a second pair of double points  $X_1, Y_1$  entails further restriction on the form of  $\varphi_r$ .

Project  $Y$  to infinity and pairs of points on  $XY$  given by  $\varphi_r = 0$  become symmetric with respect to  $X$ .

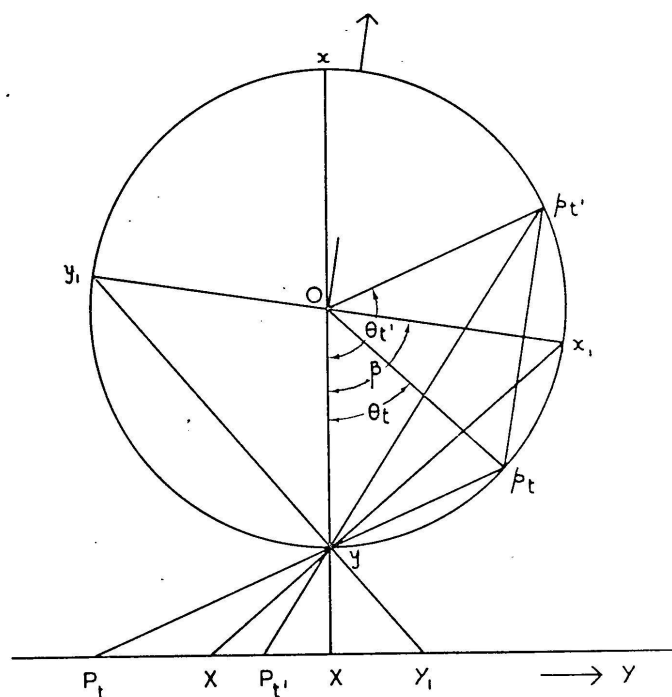


FIGURE 1.

Taking  $r = 2\rho + 1$ ,  $X_1$  is a point of  $\varphi_r$  and since there must be  $\rho$  points between  $X_1$  and  $Y_1$ ,  $X$  must separate  $X_1$  and  $Y_1$ . Taking  $r = 2\rho$  ( $\rho > 1$ ),  $X$  divides the points into two equal groups  $(\alpha)$  and  $(\beta)$  and  $X_1, Y_1$  are the double points of an involution among  $(\alpha) + (\beta)$  in some order. If  $X_1$  and  $Y_1$  are not separated by  $X$ , let them be on the side of the group  $(\alpha)$ . Since there must be  $\rho$  points of  $\varphi_r$  between  $X_1$  and  $Y_1$ , they must be the points  $(\alpha)$ . The harmonic conjugates with respect to  $X_1, Y_1$  of an ordered set of points from  $X_1$  to  $Y_1$  are external to  $X_1 Y_1$  and in the reverse order, that is they are the set  $(\beta)$ . Any point of  $(\alpha)$  is therefore paired with the same point of  $(\beta)$  in both involutions and therefore the premise is incorrect and  $X$  must separate  $X_1$  and  $Y_1$ . Let one of the intersections of the perpendicular to  $XY$  at  $X$  with the circle on  $X_1 Y_1$  as diameter be  $y$ . Since  $X$  is between  $X_1$  and  $Y_1$  this point is real. Draw any circle through  $y$  with its centre,  $O$ , on the normal  $Xy$

produced through  $y$ , and meeting the normal again in  $x$ . Denote any point of the involution by  $P_t$  where  $2l - 1, 2l$  are the suffixes of a pair of the involution with double points  $X$  and  $Y$ . In the odd degree case  $X$  is itself a point of  $\varphi_r = 0$  and may then be denoted by  $P_0$ . Projecting from  $y$  as vertex on to the circle points  $P_t$  will give points  $p_t$  which form corresponding involutions on the circle. The centre of the involution on the circle corresponding to the involution with  $X, Y$  as double points is  $Y$  at infinity.  $X_1, Y_1$  become points  $x_1, y_1$  at the ends of a diameter of the circle, and therefore the centre of this involution is at infinity in the direction perpendicular to  $x_1 y_1$ , and the joins of pairs of points of the second involution on the circle must be parallel to this direction. Let the angle between the directions of the two centres of involution which equals  $yOx_1$  be  $\beta$ . Let  $yp_t$  subtend the angle  $\theta_t$  at the centre of the circle.

(i) Even degree,  $\varphi_{2r}$  ( $r > 1$ ).

From the involution on the circle formed by the parallel chords through  $Y$

$$\begin{aligned}\theta_1 + \theta_2 &= 0 \pmod{2\pi} \\ \theta_3 + \theta_4 &= 0 \pmod{2\pi} \\ &\dots\dots\dots \\ \theta_{2r-1} + \theta_{2r} &= 0 \pmod{2\pi},\end{aligned}\tag{5}$$

and adding

$$\sum_1^{2r} \theta_t = 0 \pmod{2\pi}.$$

Pairing for the involution defined by the double points  $X_1$  and  $Y_1$  we have similarly, from parallel chords,  $r$  equations

$$\theta_t + \theta_{t'} = 2\beta \pmod{2\pi}\tag{6}$$

in which  $t, t'$  take the values  $1, 2, 3, \dots 2r$  ( $t \neq t'$ ). Hence

$$\beta = \frac{2s\pi}{2r} = \frac{s\pi}{r},$$

where  $s$  can take any one of the values  $1$  to  $r - 1$ .

If  $p_t$  is the mate of  $p_l$  in the first involution and of  $p_m$  in the second, we have

$$\begin{aligned}\theta_l + \theta_t &= 0 \pmod{2\pi}, & \theta_m + \theta_t &= 2\beta \pmod{2\pi}, \\ \therefore \theta_m - \theta_l &= 2\beta \pmod{2\pi},\end{aligned}$$

and the angle between  $yp_m$  and  $yp_l$  is the constant  $\beta$  of the second involution. Since  $\beta$  can take  $r - 1$  values, it follows that if there are two involutions among the  $2r$  points, there are  $r$  involutions.

The argument used to prove that  $X$  separates  $X_1, Y_1$  also demonstrates that between any two successive points of  $\nu_{2r}$  there cannot be more than one double point ( $X$  or  $Y$ ) of the possible hyperbolic involutions. Hence the  $r$  involutions now determined form the complete set of hyperbolic involutions among the points.

(ii) Odd degree,  $\nu_{2r+1}$ .

It has already been shown that in the odd case any pole is itself a point of the group and is a double point of an involution among the remaining points. Without any loss of generality we can therefore take  $X$  at  $P_0$  and any possible second pole  $X_1$  at  $P_1$ . Then for the first involution we have  $\theta_0 = \pi \pmod{2\pi}$  in addition to equations (5). For the second involution we have  $\theta_1 = \beta \pmod{2\pi}$  in addition to equations (6) in which  $t, t'$  take the values  $0, 2, 3 \dots 2r$ . ( $t \neq t'$ ). This finally leads to

$$\beta = \frac{2s + 1}{2r + 1} \pi,$$

where  $s$  can take any of the values  $0$  to  $2r$ . Hence if two involutions can be formed in this manner,  $2r + 1$  involutions can be so formed. Also the angle  $\angle P_0 y P$ , for any  $P$ , is a value of  $(\pi - \beta)/2$ .

There are  $2r + 1$  positions for the pole  $X$  and  $Y$  is fixed uniquely as the harmonic conjugate of  $X$  with respect to the two points of  $\nu_{2r+1}$  nearest to  $X$ .

4. We now determine the basic algebraic forms for  $\nu_r$ . Suppose  $P$  is any intersection of  $\nu_r$  with  $XY$  of the triangle  $XYZ$ , given by a factor  $mx - y = 0$  of  $\nu_r$ . Then when  $YZ$  is projected to great distance  $P$  on  $XY$  is given by  $m - y = 0$ . Hence the corresponding factor of  $\nu_r$  in the  $XYZ$  field is  $x \sin \theta - y \cos \theta$ , where  $\theta = \angle XyP$ , apart from a possible scale adjustment.

(i) It has been shown that for all polar curves,  $\rho_r = 0$ , of degree  $> 2$  with more than one pole the centres of the involutions on the circle are at great distance. A curve may also have a polar curve of degree 2,  $\rho_2 = 0$ , and hence for  $\rho_2$  we only consider possible centres of involution at great distance. Since the join of the two points on the circle has to pass through two points at great distance, the two points themselves must be at great distance and are therefore the isotropic points and

$$\rho_2 \propto x^2 + y^2.$$

Any other point on the line at great distance is a possible centre.

(ii)  $\rho_{2r}$  ( $r > 1$ ) with the corresponding points on  $XY$  all real.

Denote the angle  $XyP$ , where  $P$  is any of the points, by  $\delta$ ; then

$$\begin{aligned} \rho_{2r} &\propto \prod_0^{r-1} \left[ x^2 \sin^2 \left( \delta + \frac{s\pi}{r} \right) - y^2 \cos^2 \left( \delta + \frac{s\pi}{r} \right) \right], \\ &\propto (x^2 + y^2)^r \prod_0^{r-1} \left[ \cos \left( 2\delta + \frac{2s\pi}{r} \right) - \cos 2\Phi \right] \text{ where } \tan \Phi = y/x, \\ &\propto (x^2 + y^2)^r [\cos 2r\delta - \cos 2r\Phi], \\ &\propto (x^2 + y^2)^r \cos 2r\delta - x^{2r} \left[ 1 - \frac{2r(2r-1)}{2!} \tan^2 \Phi \right. \\ &\quad \left. + \frac{2r(2r-1)(2r-2)(2r-3)}{4!} \tan^4 \Phi \dots \right], \\ &\propto (x^2 + y^2)^r \cos 2r\delta - \left[ x^{2r} - \frac{2r(2r-1)}{2!} x^{2r-2} y^2 \right. \\ &\quad \left. + \frac{2r(2r-1)(2r-2)(2r-3)}{4!} x^{2r-4} y^4 - \dots \right]. \end{aligned}$$

(iii)  $\rho_{2r}$  ( $r > 1$ ) with the corresponding points on  $XY$  all imaginary and distinct.

The product of  $x \sin \left( \theta + \frac{s\pi}{r} \right) + y \cos \left( \theta + \frac{s\pi}{r} \right)$ , where  $\theta = p + iq$  ( $p$  and  $q$  being real and  $q \neq 0$ ), and its conjugate gives the expression

$$(x^2 + y^2) \left[ \cosh 2q - \cos \left( \Phi + p + \frac{s\pi}{r} \right) \right] / 2$$

where  $\tan \Phi = y/x$ . Giving  $s$  the values  $0, 1, \dots (r-1)$  the product of the resulting expressions is proportional to

$$\begin{aligned}
& (x^2 + y^2)^r [\cosh 2rq - \cos 2r(\Phi + p)] \\
&= (x^2 + y^2)^r \cosh 2rq - \cos 2rp \left[ x^{2r} - \frac{2r(2r-1)}{2!} x^{2r-2} y^2 + \dots \right] \\
&+ y \sin 2rp \left[ 2rx^{2r-1} - \frac{2r(2r-1)(2r-2)}{3!} x^{2r-3} y^2 + \dots \right].
\end{aligned}$$

Now  $\varphi_{2r}$  contains even powers of  $x$  only, and hence we must take  $p = t\pi/2r$ , where  $t$  is an integer, and  $\cos 2rp = \pm 1$ .

Giving  $s$  its run of values and  $t = 0$ , we get from  $x \sin\left(\theta + \frac{s\pi}{r}\right) + y \cos\left(\theta + \frac{s\pi}{r}\right)$  one set of points and a second set from their conjugates. Changing  $t$  by an even number interchanges each set within itself: changing  $t$  by an odd number interchanges the sets completely. Hence it is sufficient to take the two values 0 and 1 of  $t$ , obtaining for  $\varphi_{2r}$

$$(x^2 + y^2)^r \cosh 2rq - \left[ x^{2r} - \frac{2r(2r-1)}{2!} x^{2r-2} y^2 + \dots \right] (t = 0),$$

or

$$(x^2 + y^2)^r \cosh 2rq + \left[ x^{2r} - \frac{2r(2r-1)}{2!} x^{2r-2} y^2 + \dots \right] (t = 1).$$

(The mate of a point  $s$  in one set is given by  $s' = 2r - 4t - s$  in the other.)

An inclusive form for  $\varphi_{2r}$  is

$$a(x^2 + y^2)^r + b \left[ x^{2r} - \frac{2r(2r-1)}{2!} x^{2r-2} y^2 + \dots \right].$$

(iv)  $\varphi_{2r+1}$  with the corresponding points on  $XY$  all real and distinct.

$$\varphi_{2r+1} \propto y \prod_0^{r-1} \left[ x^2 \sin^2 \frac{(r-s)\pi}{2r+1} - y^2 \cos^2 \frac{(r-s)\pi}{2r+1} \right],$$

$$\propto y x^{2r} \prod_0^{r-1} \left[ \tan^2 \frac{(r-s)\pi}{2r+1} + \left( \frac{y}{x} \right)^2 \right],$$

$$\propto \frac{(x + iy)^{2r+1} - (x - iy)^{2r+1}}{2iy},$$

and

$$\varphi_{2r+1} \propto y \left[ x^{2r} - \frac{2r(2r-1)}{3!} x^{2r-2} y^2 + \dots \right].$$

We have so far considered only the basic forms of  $\varphi$ . We now consider the possible combinations and special forms of  $\varphi$  which may occur in a complete  $V$ .

5. If some of the intersections of  $V_r$  are real and some are imaginary (all being supposed distinct), then  $V_r$  must be replaced by  $V_s V_{r-s}$  where  $V_s$  is built up from basic  $\nu$ 's with real intersections and  $V_{r-s}$  from basic  $\nu$ 's with imaginary intersections.

The points of a  $\nu_{2r}$  will not be distinct if for some value or values of

$$\delta, s, s' (s' \neq s)$$

$$\tan (\delta + s\pi/r) = \pm \tan (\delta + s' \pi/r) ,$$

that is

$$\delta + s\pi/r = \pm (\delta + s' \pi/r) \pmod{\pi} ,$$

or

$$2r\delta = 0 \pmod{\pi} .$$

This implies that the points are all repeated and are at  $r$  alternate poles of  $\nu_{2r}$ . These points may be given by a  $(\nu_r)^2$  with  $\cos r\delta=0$ ,  $\cos 2r\delta = -1$  in the case when  $r$  is even or with  $\cos 2r\delta = +1$  when  $r$  is odd, and must then be replaced by  $\nu_{2r}$ .

Again  $\nu_2$  to any power may occur in  $V$  and for  $V_r$  we must have  $\nu_2^l V_{r-2l}$ .

The modifications required for any combination or extension of these circumstances or for peculiarities of higher orders are manifest. It can readily be shown that

$$\nu_{2p(\delta)} \nu_{2p\left(\delta+\frac{\pi}{pq}\right)} \cdots \nu_{2p\left(\delta+\frac{(q-1)\pi}{pq}\right)} = \nu_{2pq(\delta)} ,$$

where the portion of the suffix in brackets denotes the value of " $\delta$ " for the attached  $\nu_{2p}$  and defines the position of the points of the involution. The corresponding factor of  $V$  is then  $\nu_{2pq}$ .

It can also be shown that

$$\nu_{(2p+1)} \nu_{2(2p+1)\left\{\pi/(2p+1)(2r+1)\right\}} \cdots \nu_{2(2p+1)\left\{r\pi/(2p+1)(2r+1)\right\}} = \nu_{(2p+1)(2r+1)} .$$

If  $p, q$  are odd and  $p$  is prime the expressions  $\nu_{pq}$  and  $\nu_{2mp}$  cannot have a common factor unless the  $p$  poles of  $\nu_p$  are points of  $\nu_{2mp}$ .

$V_r$  and  $\nu_r$  will be referred to as of order  $r$  and are homogeneous of degree  $r$  in  $x$  and  $y$ .

6. We now consider the assemblage of terms which form  $f(x, y, z), \Sigma \alpha_r V_r z^{N-r}$ , where the  $\alpha$ 's are constants, and the  $V$ 's

are of the forms previously specified. Possible involutions determined by each  $V_r$  may be displayed on a circle, and the only possible poles for the complete curve are given by the common involution centres of all  $\varphi$ 's which occur. The isotropic points given by  $\varphi_2 = 0$  belong to every involution. The number of possible poles is therefore equal to the highest common factor,  $p$ , of the orders of the  $\varphi$ 's, not including  $\varphi_2$  and its powers. This implies that for any  $p$  the form of  $V_N$  imposes conditions on the existence or the form of any  $V_{N-k}$ . If  $p$  is even the number of involutions is  $p/2$  and both double points of each involution are possible poles, and the conjugate line of any such point meets  $XY$  in a possible pole. If  $p$  is odd the number of involutions is  $p$ , and only one double point of each involution is a possible pole: the intersection of the conjugate line of such a point with  $XY$  is not a pole. A possible pole will actually be a pole if its conjugate line passes through  $Z$ .

The form of the function can be slightly modified by a real scale change.

7. Consider a curve having a pole at  $X$  and with a multiplicity of order  $k \geq 0$  at  $X$ . From the basic forms (1) and (2) if the curve is of even degree,  $k$  must be even and if the curve is of odd degree,  $k$  must be odd. In both cases  $N - k$  is therefore even. We will also suppose that there are further possible poles on  $XY$ . Since  $X$  is a  $k$ -ple point no powers of  $x$  greater than  $N - k$  may occur in the equation of the curve and this power must occur.

$V_{N-k+t}$ ,  $t$  0 to  $k$ , if it occurs, must contain  $y$  to the power  $t$  at least as a factor and "at least" must be replaced by exactly for one or more values of  $t$ . A  $V$  which contains  $y^t$  as a factor is necessarily the product of  $a$   $\varphi$ 's of odd order  $2m_s + 1$ ,  $s$  1 to  $a$ , and  $b$   $\varphi$ 's of order  $2(2n_s + 1)$ ,  $s$  1 to  $b$ ,  $\cos 2(2n + 1)\delta = 1$ , where  $a + 2b = t$ . The total number of poles (actual and possible) on  $XY$  is a factor of  $2m_s + 1$  and  $2(2n_s + 1)$  and hence this number may be taken as  $p$  or  $2p$  where  $p$  is odd. From paragraph 5 it follows that  $\varphi_p$  is a single factor of each of the  $a$   $\varphi$ 's of odd order and a repeated factor of each of the  $b$   $\varphi$ 's of even order, and the corresponding  $V$  is the product of  $(\varphi_p)^t$  and an expression  $\omega$ . We can now write the equation of the

curve in the form

$$f(x, y, z) \equiv \alpha_0 z^N + \alpha_2 z^{N-2} \varphi_2 + \dots + \alpha_{N-k-1} z^{k+1} V_{N-k-1} \\ + \sum_{t=0}^{t=k} \alpha_{N-k+t} z^{k-t} (\varphi_p)^{t+\tau} \omega_t = 0,$$

where  $\tau$  is zero or a positive integer for each term. A term containing  $x^{N-k}$  is obtained when  $\tau$  is zero and must occur at least once.

If  $P$  is any possible pole on  $XY$  and  $P'$  its conjugate phase point, then the harmonic conjugate of  $X$  with respect to  $P, P'$  is also a possible pole and a  $k$ -ple point. Hence when the number of possible poles on  $XY$  is odd every such point is a  $k$ -ple point and when the number is even every alternate point is a  $k$ -ple point. In each case these possible poles (at  $k$ -ple points) are the intersections of  $XY$  with  $\varphi_p = 0$ . Let  $X' (x' : y' : 0)$  be such a point. The tangents at  $X'$  are given by

$$\left( X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z} \right)^k f(x, y, z) = 0,$$

and these together with the conjugate line of  $X'$  are given by

$$\left( X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z} \right)^{k+1} f(x, y, z) = 0,$$

where  $x, y, z$ , are replaced by  $x', y', 0$  after differentiation and  $X, Y, Z$ , are here used to represent the current coordinates. Now

$$\left( X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z} \right)^k z^{k-t} (\varphi_p)^{t+\tau} \omega_t \\ = \frac{k!}{t!(k-t)!} Z^{k-t} (k-t)! \left( X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} \right)^t (\varphi_p)^{t+\tau} \omega_t,$$

on putting  $z = 0$  after differentiation. The expression

$$\left( X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} \right)^t (\varphi_p)^{t+\tau} \omega_t$$

vanishes except when  $\tau = 0$  since  $\varphi_p = 0$  when  $x = x', y = y'$ . The tangent form is obtained from those values of  $t$  for which the corresponding  $\tau$ 's are zero. If  $t'$  is the least of these values,

the highest power of  $Z$  is  $k - t'$  and the tangents consist of  $k - t'$  lines not through  $Z$  and  $X'Z$  counted  $t'$  times.

Now  $V_{N-k-1}$  is of odd order, hence if the number of possible poles is odd the term  $\alpha_{N-k-1} z^{k+1} V_{N-k-1}$  either contains  $\varphi_p$  as a factor or  $\alpha_{N-k-1} = 0$ , and if the number is even  $\alpha_{N-k-1}$  must be zero. In either case the term does not appear after the substitution of  $x_1, y_1$  for  $x, y$  and therefore makes no contribution to the tangent and conjugate line form. We now obtain

$$\frac{(k+1)!}{(t+1)!(k-t)!} Z^{k-t} (k-t)! \left( X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} \right)^{t+1} (\varphi_p)^{t+\tau} \omega_t,$$

leading in exactly the same way as before to  $k - t'$  lines not through  $Z$  and  $X'Z$  counted  $t'$  times and a line also through  $Z$ . This last line must be the conjugate line of  $X'$ . It follows that the conjugate lines of all the possible poles on  $XY$  pass through  $Z$ , a point determined previously by the conjugate lines of  $X$  and a specific pole  $X_1$ . Hence all the possible poles are actual poles.

Special forms of the  $V$ 's can lead to multiple points on  $XY$  which are not poles.

**8.** The general problem of the total number of poles or of their distribution has not been solved. From the degeneracy of the first polar curve with respect to any pole  $P$  into a curve of degree  $N - 2$  and the conjugate line, the Steinerian must have a multiplicity of order  $N - 2$ , or equivalent singularity, at  $P$ . There is therefore, in general, an immediate crude finite limit to the number of poles. When  $N$  is odd another limitation is provided by the number of real inflexions. It has been possible however to obtain some detailed information concerning properties and numbers for certain types of curve.

We show first that any curve of even degree whose equation is symmetric in  $x^2, y^2, z^2$  has nine poles on the sides of the triangle of reference. Its equation can be written as

$$\sum a_{rst} s_1^r s_2^s s_3^t = 0,$$

where

$$s_1 = x^2 + y^2 + z^2, \quad s_2 = y^2 z^2 + z^2 x^2 + x^2 y^2, \quad s_3 = x^2 y^2 z^2.$$

It clearly has poles at  $X, Y, Z$ .

Making the transformation

$$x' = x, \quad y' = y + z, \quad z' = y - z,$$

leads to an equation of even degree in  $x', y', z'$ , and the curve therefore has poles at  $X'$  (which is  $X$ ),  $Y'$  and  $Z'$ , the vertices of the new triangle of reference. Referred to the original triangle the curve has poles at  $(0:1:1)$  and  $(0:1:-1)$ . Similarly there are poles at  $(1:0:1)$ ,  $(-1:0:1)$ ,  $(-1:1:0)$  and  $(1:1:0)$ .

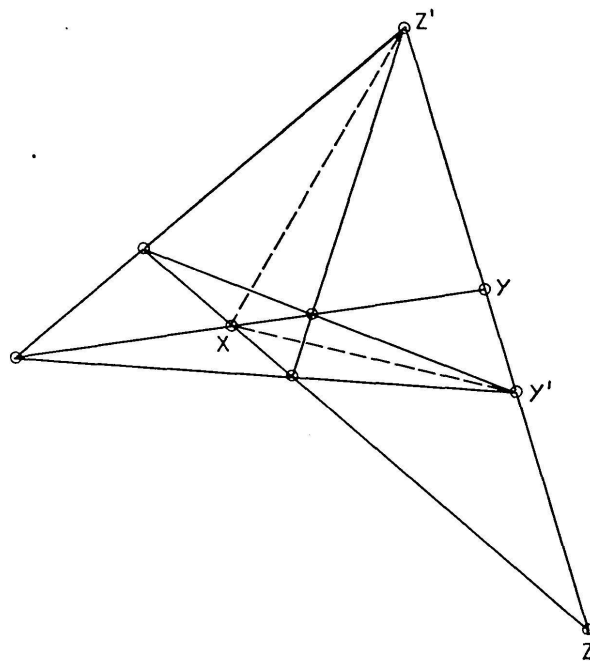


FIGURE 2.

These six new poles are collinear in threes and the lines through them form a quadrilateral of which  $XYZ$  is the harmonic triangle. Any triangle of type  $XY'Z'$  is such that each side is the polar line of the opposite vertex and referred to such a triangle the curve has poles at its vertices and at points not on the sides.

We have established that if there are  $r$  poles on a straight line they can be transposed on to a circle to form an equispaced system of points. If  $r$  is even and  $p$  is the transpose of a pole  $P$ , the point  $p'$  diametrically opposite to  $p$  is also the transpose of a pole  $P'$ . If  $r$  is odd, the point  $p'$  does not correspond to a pole but to a point  $P'$  which is the double point corresponding to  $P$  of an involution formed by the intersections of the curve with the given line. In either case the conjugate line of  $P$  passes through  $P'$  and that of  $P'$  through  $P$ . Any such point,

$P$  or  $P'$ , will be referred to as a phase point, and if two phase points are such that the conjugate line of each passes through the other they will be termed conjugate phase points. From consideration of the circle it follows that if  $Q$  is a phase point, not necessarily a pole, then its harmonic conjugate with respect to  $P, P'$  is also a phase point such that both points are poles or neither are poles. It is obvious for the circle and therefore for the line that when  $r$  is odd the poles alternate with the remaining phase points.

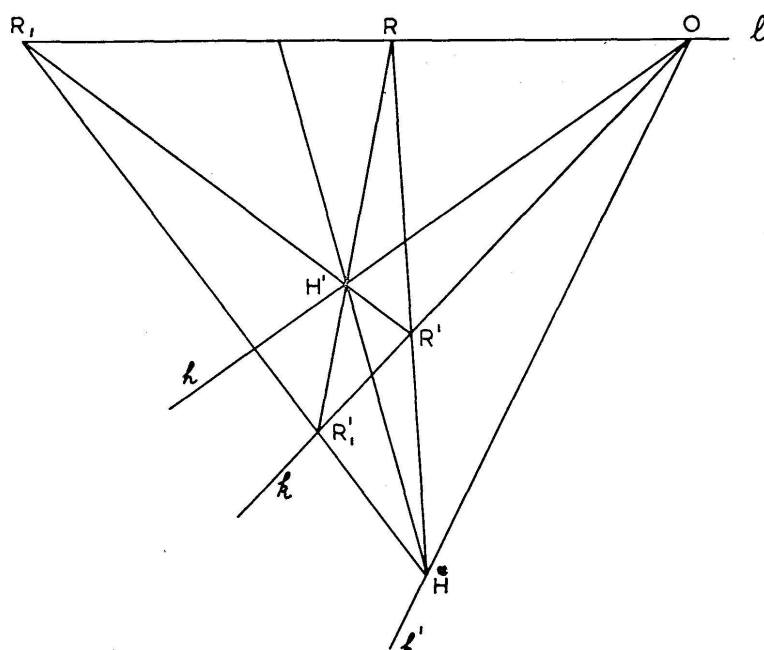


FIGURE 3.

Suppose there are  $r$  poles,  $R$ , on a line  $l$  and that  $H$  is a pole not on the line (see fig. 3). Let  $h$  be the conjugate line of  $H$  meeting  $l$  in  $O$  and let  $k$  be the harmonic conjugate of  $l$  with respect to  $OH$  and  $h$ . Then from the previous theorem the join of  $H$  to any pole  $R$  meets  $k$  in a point  $R'$  which is also a pole, and hence there are exactly  $r$  poles on  $k$  since the process is reversible. If  $O$  is a pole, its conjugate line will pass through  $H$  and hence meets  $h$  in a pole  $H'$ . The conjugate line of  $H'$  is  $OH$  and its  $k$  line coincides with the  $k$  line of  $H$ .  $HH'$  will pass through a phase point on  $l$  which will be a pole only if  $r$  is even.

Suppose  $P_1 P_2 P_3 P_4 P_5$  is a regular pentagon with circumcentre  $Z$  and let the regular pentagon formed by the joins of

alternate vertices be  $Q_1 Q_2 Q_3 Q_4 Q_5$  with  $P_r ZQ_r$  collinear. Let the point at great distance on the side opposite  $P_r$  be  $X_r$ , and the point at great distance on  $P_r Q_r$  be  $Y_r$ . Suppose a curve can be drawn symmetric about the five lines  $ZP_r$  and with a pole at  $P_1$  with corresponding conjugate line  $Q_1 X_1$ . By symmetry all the  $P_r$ 's are poles. The  $k$ -line of  $P_2 Q_5 Q_1 P_4 X_3$  with respect to the pole  $P_1$  is  $X_4 P_3 Q_1 Q_2 P_5$ , the joins of corresponding points passing through  $P_1$ . Hence  $X_4, Q_5, Q_2, X_3$  are poles and by symmetry all  $Q_r$ 's and  $X_r$ 's are poles, giving a closed system of poles. Since  $P_1 Q_1 X_1$  is a self polar triangle with all the vertices poles, the curve is of even degree. The configuration has poles not on the sides of a basic triangle but no algebraic curve has been found to satisfy the primary conditions.

The joins of any pair of conjugate phase points on  $XY$  to  $y$  (fig. 1) are at right angles and therefore after projecting  $XY$  to great distance from  $y$ , the pencil of  $2q$  lines joining  $Z$  to the phase points on  $XY$  forms an orthogonal involution. Calling the phase points taken in ordered sequence  $Z_0, Z_1, Z_2, \dots, Z_{2q-1}$ , the lines  $ZZ_r, ZZ_{q+r}$  are at right angles. Since the conjugate of  $Z_r$  with respect to  $Z_0$  and  $Z_q$  is  $Z_{2q-r}$ , the lines  $ZZ_r, ZZ_{2q-r}$  are equally inclined to  $ZZ_0$  and  $ZZ_q$ . All suffixes are mod  $2q$ . The  $2q$  lines therefore form an equi-spaced system, the angle between any two successive lines being  $\pi/2q$ . Let  $L$  be any point on the curve and let  $Z_{q+r}$  be a pole. If  $L'$  is the image of  $L$  in  $ZZ_r$ , the conjugate line of  $Z_{q+r}$ , then  $ZL, ZL'$  are harmonically separated by  $ZZ_{q+r}, ZZ_r$  and  $L'$  also lies on the curve. The curve is therefore now symmetrical about the conjugate lines of all the poles at great distance. If  $H$  is any other phase point, necessarily finite, its kaleidoscopic images in these conjugate lines are, from the symmetry, also phase points and of the same type.