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THE MATHEMATICS EDUCATION OF YOUTH A COMPARATIVE STUDY

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What kind and how much instruction in Mathematics is received by the youth of the world up to the age of fifteen years ? Sixteen countries[1]) have supplied to the writer a summary for their country on this topic.

To attempt to make comparisons or contrasts of the programs of the different countries is exceedingly difficult and would serve no useful purpose. This report will attempt to give an overall viewpoint of mathematics instruction in all the countries concerned, with regard to the following phases: (1) the material or subject matter included in the program; (2) The school organization and the sequential arrangement of the subject matter by years of instruction either grades I through IX or X; or ages 6 years to, but not including 16 years; also the time allotted to mathematics instruction; (3) The selection, promotion, and segregation of pupils into special classes; particularly those classes designated as preparatory to University entrance; (4) The methods of instruction with special reference to desired goals of pupil achievement; (5) The Preparation of the teachers of mathematics; (6) The systems of examinations; (7) The directions and trends that instruction is taking with regard to philosophical, cultural and psychological aspects of learning.

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[1]) In each of the sixteen countries, during the years 1955-1958 a sub-committee of the International Commission of Mathematics Instruction, has investigated for their particular country the program in mathematics education for children from the age six through fifteen years. This article is a synthesis of all of these studies. The Countries are: Austria, Canada, Finland, France, Germany, Great Britain, Hungary, India, Italy, Japan, Jugoslavia, Netherlands, Norway, Russia, Sweden, and United States of America.

The Subject Matter.

Mathematicians are prone to think of mathematics and school arithmetic as two separate and only slightly related disciplines. Thus school arithmetic, or Rechnung, or computation is a set of rules and mechanical operations to be learned in the early years of schooling, while arithmetic, related to Algebra is the theory of numbers and is the beginning of the study of mathematics. This point of view was present in many of the reports, but not all. The subject matter thus studied in these years can be classified as Number or Computation, Algebra, Geometry, and Numerical Trigonometry.

Under Number the following topics are universally studied. Numbers as counting numbers, first to five, then ten, then 20, then 100, then 1000, and up to 10,000,000. The decimal system of notation; the Roman numerals to thousand. The addition and subtraction of whole numbers with and without carrying or borrowing. In many countries the Austrian Method[2]) of subtraction is mandatory right from the start.

Multiplication tables are memorized and include all combinations up to 10×10, some countries demanding up to 12×12, and one country including all combinations to 20×20. The multiplication of whole numbers begins with a single digit multiplier, and then two and three digit multipliers, preceded by multiplication by powers of 10. Division, considered the most difficult process is the last to be taught beginning with divisors of 2, 3, 5 and 10; and finally 2 and 3 digit divisors. Once the process has been taught there are applications to simple problems of everyday life.

The meaning of simple fractions as equal parts of a whole such as 1/2, 1/3, or 1/5 is introduced early, but actual operations with fractions (called common in America and often referred to as vulgar elsewhere) is delayed until all operations with the whole numbers have been taught. The first operation consists

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2) The Austrian method uses the principle of adding the same number to the minuend and subtrahend. Thus in 42-18, ten is added to the 2 in 42 to give 40 and 12, and ten is also added to the one ten in 18 to give 28. Then the substracter says " 8 and what gives 12 ? " Answer " 4 ". Then " 2 and what gives 4 ? " Answer " 2 ". The difference is 24.

of changing (so called reducing) of fractions to the same denominator, and in quite a few countries the method is based on that of finding a Least Common Multiple. Then the four operations with fractions are taught, including operations with mixed numbers. Finding the Greatest Common Factor is also stressed in many countries.

Decimal fractions, or better fractions written in decimal notation are usually delayed until the 5th or 6th year of instruction. Before this, however, countries using the metric system or a decimal money system have included numbers written with a decimal point (or comma). However the four operations on mixed decimals, that is, numbers with whole and fractional parts written in decimal notation, is taught in all countries. The equivalence of common fractions to decimal fractions and making the transformation from one form to the other culminates this aspect of computation.

Along with the whole numbers, systems of measures are introduced and gradually extended so that operations can be performed with them. The measures include the local national ones, as well as the metric system, but the latter is delayed until the 7th of 8th school year in all countries that do not use it as their basic system. The measures include those for length, area, volume, capacity, weight, money and time. Exchange of money is taught in most countries.

Percentage, as a special topic is taught in all countries. Much emphasis is put on this topic and applications are made to simple and compound interest, discount, chain discounts, profit and loss, commission, borrowing, instalment purchases, stocks and bonds, and other business affairs. The application of arithmetic to business and daily life problems seems to have grown in amount and stress in all countries during the past few decades.

Ratio and proportion are taught universally, a very common part of this work being the rule of three. A few countries still teach alligation simple and alligation compound[3]) with

[3]) Alligation refers to the rules of finding the proportions of various ingredients in a mixture to meet specified requirements on cost or strength. The word disappeared from American text books more than 40 years ago.

applications to mixtures, work problems, and the like. There are also applications to scale drawing and map-reading. The unitary method was commonly mentioned as required work.

All countries teach the concept of average, whereby is meant finding the arithmetic mean of set of numbers. A few countries teaching divisibility by 2, 3, 5, and 9, are using divisibility by 9 as a check on the operations of multiplication and division. Prime numbers receive scant or no attention. It is noteworthy that no country reported teaching numeration to any other base than 10, nor did any country report teaching the fundamental laws of arithmetic (as a Ring) in developing the rules of operation. Thus it must be generally concluded that the teaching of arithmetic is looked upon as the development of a tool with many tricks and manipulations, that are to be used in special types of practical problems. Theoretical considerations are at a minimum. Reasoning is not demanded, but skill in manipulating is of the essence.

The work in Geometry begins with the study of the ruler, using it to mark off and to measure distances. The common geometric figures, square, rectangle, triangle and circle are illustrated by drawings and observed in nature. In all countries the first concepts taught are that of perimeter and area of the square and rectangle. The first approach to area is through the use of squared paper, where the squares in the interior of the figure are counted. The continuation of the study of geometry may be described as a gradual approach to demonstrative methods by way of concrete measurements, intuitive study of properties of plane figures and solids, simple deductions, and finally a study of the nature of proof. In the intuitive approach there is covered the measurement of lines and angles, the perimeter and area of plane figures including the circle, the areas and volumes of the common solids — cube, rectangular prism, regular prisms, right circular cylinder, regular pyramid, right circular cone, and the sphere. The results are determined by practical means and the use of models, and then generalized by formulas. The work on geometry next concerns itself with parallel lines, perpendicular lines, angle bisectors, congruent triangles and rectangles and is accompanied by much construction work with compasses and rulers, as well as set-squares, triangles and protractors. Many of the usual theorems of plane geometry are thus evolved as facts or relation without any formal deductive system. The third aspect of intuitive geometry may be called relational, and here there is studied the base angles of an isosceles triangle, complementary, supplementary and vertical angles, the relation of sizes of angles to the opposite side of a triangle, the sum of the angles of a triangle and the Pythagorean relationship for a right triangle. At this place, in instruction practically all countries report the finding of squares and square roots of numbers, both by numerical methods and by the use of tables. The use of Newton's method of division and averaging the quotient and divisor is the only method reported, the Euclidean method having gone into discard at this stage of learning.

Along with the Pythagorean theorem, ratio and proportion are introduced into geometry via similar triangles and then the sine, cosine and tangent ratios are defined for angles between 0° and 90°. These ratios are used to find sides and angles of right triangles from given numerical values for the other sides and angles. Applications are made of the trigonometric ratios and the Pythagorean theorem to practical problems, especially those of simple surveying and navigation. In this study such instruments as the clinometer, hypsometer, transit, sextant, and angle mirror are used. The work is accompanied by drawing to scale and the making of simple plans.

In most countries, but not all, by the age of 14 years or in grades 8 and 9, the children have been introduced to the axioms, the use of definitions, the simple syllogism and a deductive chain of theorems, that is, the study of Euclidean deductive plane geometry. The amount of deductive geometry taught by age 15 years (grade 9) varies from a study of only congruence and parallelism to that of completion of all theorems on rectilinear figures, the circle and angle measurement, similarity area, and the regular polygons. There is far more variation in the amount of geometry studied in the various countries, than there is in the content of algebra.

The approach to algebra is usually through the generalization of arithmetic by the use of letters for numbers. Thus in many countries arithmetic means introductory or manipulative algebra. Letters are used with fundamental operations and then equations are introduced. Other countries approach the simple equation through the study of formulas, where letters are used to formulate general rules. A third approach is to begin algebra by the study of positive and negative numbers, develop the laws of these numbers and formulate them with letters leading to identities such as

$$x + y = y + x$$

and

$$a\,(b + c) = ab + ac\,.$$

The stress in the algebra study at age 14 or 15 is on the solution of equations, first the simple equation in one unknown, then two equations in two unknowns (or three unknowns), and finally the quadratic equation. In so far as the reports show, the emphasis is on tricks and formulas and not on proof. The word theorem or proposition is a rarity in the first study of algebra. The remaining study of algebra is given over to special products and factoring; the linear function; direct and inverse variation; and the graph of the linear and quadratic function. Only two countries report more study including such topics as Highest Common Factor, Lowest Common Multiple, involution and evolution; fractional indices; surds; the function $y = \sqrt{x}$; and the derivative of a polynomial.

This, then is the picture of what the pupil has been taught. What does he really know ? This is hard to tell, but it can be said that the 15 year old in all countries, who has continued his study of mathematics through the first 9 or 10 years of school can compute in a mature manner with the positive rational numbers, in a decimal system of notation, even though he cannot rationalize what he does; he has a fairly useful and practical knowledge of geometry with respect to mensuration and common relationships; and he can manipulate algebraic expressions and solve equations and problems in a structureless system of algebra. He can make simple deductions, but his entire concept of proof, if any, is limited to that of theorems in geometry. He really does not know what mathematics is, or how it is applied, but he has a large body of information, upon which, if he is inclined or interested, a study of mathematics can be built in the ages 16 years to 21 years. The whole program, the world over, is overloaded with " doing " and it would appear that a reformation of this program with emphasis on " reasoning " and an elimination of much useless and extraneous busy work could enliven the subject and leave the 15 year old with a much clearer and stronger picture of what mathematics study really is.

School Organization and Time Allotted to Mathematics Study.

One thing is certain, and that is that the school organization and selection of students is unique in each of the countries having compulsory (and free) education. The starting age for grade 1, or first year of formal schooling varies from 5 years to 7 years of age. Thus by the age of 15 years, youth in the several countries have had from 8 to 10 years of schooling. But the number of years of instruction is modified somewhat by the fact that the number of clock-hours of instruction per week, and the number of weeks per year vary greatly from country to country. Thus the total allotted time for mathematics instruction compared with the entire instructional time in a given country varies from 20% to 9%. Statistics are boring, so here follow a few sample programs and time allottments that are indicative of the highest, lowest, and median of the countries reporting.

In all countries, the first four years of instruction is given in an elementary or folk-school, which all children regardless of ability or social origin may attend. The classes are taught by a teacher who teaches all subjects, that is, there is no special teacher for mathematics or the other branches of learning. The only exception to this statement is the use of special teachers fort art and music in these four grades. Evidently most countries assume that art and music are special talents that all teachers cannot learn to teach, but that this is not the case with language, mathematics, and history.

All countries continue the general elementary education under one teacher up to the end of the sixth, seventh, or eighth grades. However, in many cases there is a splitting of those who " can " from those who " cannot " into separate schools and separate programs beginning at the fifth, sixth or seventh school year. Only two countries, Canada and the U.S.A., maintain a common school throughout the first eight years of study. There is quite a general agreement on compulsory education of all children for 8 years, but this is not strictly adhered to. The trend is to increase the number of years of required attendance at school. In all countries mathematics is a required subject of instruction throughout these 8 years. Where the capable pupils are separated into special schools, mathematics study for these pupils is required every year up to the age of 15 years (and beyond).

In Russia, formal schooling begins at age 7. During the first seven years of schooling (up to the age of 15 years), there are 6 lessons in mathematics every week for 33 weeks of the year. That is 198 lessons per year and this comprises 20% of all the teaching time. During the first five years all the work is on arithmetic; during the sixth year, 2 lessons each on algebra, geometry and arithmetic, and during the seventh year, 2 lessons on geometry, 4 on algebra. During the first four years one teacher gives all the instruction, after grade four all teaching is done by specialists. The instruction in all seven years is compulsory for all students with no separation into special classes.

In France, the elementary school begins at age 6 and runs for 5 years. During the first three years $3\frac{3}{4}$ hours per week are devoted to the study of mathematics which is 12% of the total teaching time. During the 4th and 5th years, the time is increased to 5 hours per week or $16\frac{2}{3}$% of the teaching time. However, in the next four years, where the students study in separate classes according to ability and future aims, the weekly study of mathematics is reduced to 2½ hours, soon to be increased to 3 hours or only 10% of the teaching time.

In the German schools, for the first 9 years of study of those entering the Mittelschule or Gymnasium, mathematics instruction takes 15% of the teaching time, and this same percent applies to the countries Norway, England, and Sweden. In Japan, since the reorganization of its schools after the war, there is a six year common elementary school and during the first four years only 10% of the teaching time is given to mathematics. This is increased slightly during the fifth and sixth year, and there is a trend to increase the amount of time given to the study of mathematics. Generally it appears reasonable to say that mathematics instruction of youth up to the age of 15 years occupies about one-seventh of all the teaching time, and that this ratio does not increase, rather it decreases as youth continue studying beyond age 15, unless they go into specialized study of science. How few do this will be shown later.

When pupils leave the common elementary school to go to special schools to take special courses such as are offered in the gymnasium, Realskole, Grammar School, High School — College Preparatory Course and the like, they must be assigned either by directive or by choice. In most cases the selection is made by results of examinations, in which an examination in mathematics is an important part. The other areas examined are usually the national language and history. The examination is usually written, but in some cases, consists also of an added oral examination. Some typical examples are the following: In Finland, at the end of the fourth grade, all pupils seeking admission to the secondary school sit for an examination. Recently about one-half of all school children took this examination; 73% of these passed the examination, but due to lack of space only 60% of these were admitted. Hence, beginning at grade 5 (age 11-12) only 30% of the youth of Finland have a possibility of professional careers. What part of this 30% succeed on getting to college is hard to say, but by comparison with other countries certainly at the very most ¼ or about 7% of all the youth.

In England, a similar examination procedure is necessary to enter the Grammar school and takes place at age 11 years. The percent of all the pupils passing the examinations varies throughout the country but averages about 20%. In Scotland this examination is postponed to age 12 years. In India, where the curriculum is very advanced, there is an examination every year for passing to the next grade. About 50% fail these examinations every year, and these examinations are now regarded as the worst feature of their educational system. Similarly France, Germany and Denmark have entrance examinations at a very early age (10 or 11 years) for admission to the schools preparing for University attendance. Norway has an examination at the end of the seventh school year, and Jugoslavia at the end of the eighth school year. In France, selection is made at the end of the 5th school year, by the use of the pupils' previous marks, but the pupil may apply to take a written examination if his grades do not permit his selection. In the Netherlands, there is an informal examination in mathematics at the end of each year in which about 10% fail and must repeat the year's work. But at the end of the sixth year there is a very severe examination and only a very small percent of the total school population (less than 15%) is admitted to the Gymnasium or Higher Burgher School.

Only Canada and the U.S.A. have no selective examinations. However, at the end of grade eight or nine, the pupils are advised to take programs adapted to their inherent abilities, and their probable life work when they finish school, but it must be stressed that the choice made is voluntary on the part of the pupil and his family.

The segregation of pupils at too early an age must be looked upon with some grave reservations as to its consequence in this day and age. No matter what is said about transfer, it is the universal rule, that *once ruled out*, a pupil rarely has opportunity to cross to the better track. During the ages of 11 to 14 years, boys and girls are undergoing physical changes that have real psychological implications. It would appear that a choice at age 14 years would be far more significant of real ability and opportunity than at age 11. In Germany, and the same would hold for many other countries, about 25% of fifth grade students enter the Gymnasium, but only ¼ of these eventually finish the full 9 years. Of those completing the Arbitur, only 30% are in the Scientific line and this means that about 2 out of every 100 pupils completing grade 5, study a program in mathematics throughout the next 9 years so as to enter a University to major in mathematics or science. In this day and age, this is too small a number for the needs of our society. Perhaps the grave shortage of mathematicians and teachers of mathematics, may in some measure be traced to too highly rigid selective processes *at too early an age.*

The Teachers of Mathematics.

In all countries, a shortage of teachers of mathematics is occurring and in many countries it has reached a critical state. There is bound to be a relaxing of certification qualifications in the years immediately ahead, so that any report on teachers of mathematics must be based on stated requirements rather than those actually achieved. Generally teachers of grades 1 to 4, and in the elementary school from grades 4 on to 7, 8, or 9, have graduated from a secondary school in the academic, that is college preparatory line, and have attended a teacher training college or pedagogical academy for a period of from two to four years. They have had courses in teaching arithmetic, but have studied very little or no mathematics after entering their teacher training program.

Teachers in the program from grades 5 to 8 (or 9) may have been trained in teacher colleges, but in addition have also continued their education in special subjects and taken examinations in these subjects before being certified. Teachers above the eighth year of study are University graduates with a major or minor in mathematics. They are specialists in their field. In general, the training of teachers of mathematics as specialists in instruction for the fifth to the tenth grades in European countries, includes the study of far more mathematics, than in Canada or the U.S.A.

Philosophy of Instruction.

The aims of mathematical instruction, the methods and materials used in instruction, and the proposed reforms and trends in the various countries show great contrasts as well as universal movements. These pedagogical phases are reflected in two outstanding view points which we can label (1) mathematics for the better life, i.e. for its intrinsic value, or for its own sake and (2) mathematics for a better living, i.e. for its application to science, technology, and social problems that will result in more efficient practical day by day living. Thus, for one country the goal of mathematics instruction is stated as follows:

" The pupils must learn to compute knowingly, rationally, and quickly both in written and oral work; to use arithmetic to solve practical problems and to answer questions; to develop logical thinking, initiative and creative powers; to acquire logical and systematic concepts of space; to solve practical exercises essential for use in polytecnical study."

Another country reports that the purpose of teaching mathematics is:

" To develop basic knowledge and lay the foundation for further study. In arithmetic, problems are preferentially applied to the industrial and technical sphere; in algebra, the start is with concrete problems and principles that can be applied to the solution of equations; operations in algebra are by functions — first experimentally and numerically — followed by logical elementary proofs of theorems. This simplifies the transition to deductive geometry. In the elementary school inductive methods are used, namely observation and description, making of models, abstraction of properties, formulation of definitions and laws and finally logical models, the latter however not in perfection."

The opinion that teaching in mathematics must begin with concrete physical things, the study of which, by inductive methods, leads to certain abstractions from which a mathematical system can be built, prevails all over the world. That arithmetic could be taught without the use of rather elaborate equipment — beads, abaci, number charts, colored pegs, Montessori materials, and so on ad infinitum — is an alien thesis to present day elementary mathematics instruction. So it appears that the mathematical education of youth must reflect the historical development of mathematics — i.e.: there must be a period of informal experience — then a fumbling of explaining the environment — out of which fundamental concepts emerge, are clarified and refined — then a formalization of the mathematics with memorization and application.

In the spirit of this theory the instruction during the first four or six years is guided by a predominance of psychology of learning in which more attention is given to the child's learning ability and his social needs than to the subject matter. In many countries however, just before the examination for admittance to the secondary program, the only social need becomes the learning of sufficient skills and tricks for passing the examination. Thereafter, in the secondary school the subject matter of mathematics and its gradual axiomatizing becomes the predominant factor. Since the child's ability to learn is not considered, a large and continuous rate of failure persists in the secondary school.

A glance at most textbooks for the ages 6 to 15 years reveals the startling fact that there is a tremendous amount of repetition of previous material, repeated in the same dull spirit as originally presented several years prior. Evidently no one expects a child to have learned and remembered the material taught in previous years. It is rather encouraging then to see a few countries taking a decided stand against this stultifying method by saying: " The work of the secondary school is not to repeat the study of the elementary school but to base its teaching and build new knowledge on the mathematics previously learned." If our students were expected to know what they had been taught — the material on which they had passed examinations the year before, there might be a resurrection of student interest in mathematical study, and in teaching methods, that could well border on the miraculous.

Before we turn to some promising trends in the teaching of elementary arithmetic, there is one instructional feature that all countries insist upon — namely Mental Arithmetic. However, the concept of mental arithmetic is not the same for all countries. Whatever it is, almost universally the use and stress on mental arithmetic begins in the second year of elementary school and continues, usually with daily (or at least periodic) drills, right through the age of 15 years. The one concept of mental arithmetic that is predominant is that of rapid calculation without the use of paper or pencil. Short cuts and tricks are learned (sometimes rationalized) but the purpose is to save time for later mathematics. Speed is of the essence and of course accuracy is demanded. The second concept extends beyond computation, to problem solving, allowing the use of the basic structure of the decimal system and its laws of operation, for the mental estimation, approximation, and exact solution of problems as well as for checking. Its emphasis is on thinking — reasoning, and understanding and not on speed. This concept offers power to the initiative and creativity of pupils learning, as well as interest and challenge in the subject, and it in no way deters from speed for those pupils who are capable.

Some Trends.

All countries are engaged in studying their mathematics education. A few countries are engaged in systematic experiment, but most study is made by scattered efforts of a few leaders or interested persons. Whether by parental pressure, experiment, or changing cultural patterns, there has been a gradual shift from mere rote — manipulative teaching of arithmetic, through complicated computational exercises, to the teaching by rationalization of the fundamental concepts and laws underlying the operations on number, including the decimal system of notation. Such a shift can be looked upon only with favor by those interested in the mathematical knowledge of our future society.

The result is that the work of the first four or six years is no longer regarded as reckoning or arithmetic, but as mathematics and is being labelled as such in the schools. The one drawback to the rapid promotion of this ' rational ' point of view is the lack of knowledge of the elementary school teacher on the nature of the mathematical structure of arithmetic. It appears that to date no countries are making any determined effort to improve this state of affairs.

Along with this shift in emphasis, there has also been a decrease in the home work demands upon the pupils, and a relaxing of requirements by delaying the introduction of newer topics. Thus the operations with fractions is now a fifth or sixth year instruction topic in all but a few countries; and these fractions are frequently limited to simple denominators, having fairly easily discovered Least Common Multiples. The elementary school is being extended from the fourth to the sixth or higher grade and the trend is strong to have all children undertake the same mathematics study throughout a period of eight years. The *same study* seems to make sense, but if this implies the same amount, at the same rate of teaching, then it contradicts all that we know of the great individual difference in ability and in mental growth of children in all countries.

The following trends exhibited by a few individual countries are merely noted: (1) Delaying the more logical aspects of plane geometry and preparing for it, by introducing theorems and deductions in the initial study of algebra; (2) Re-introducing rigid motion into the study of geometry, i.e. the intuitive approach to the preservation of metric properties by rotation, translation, and reflections; (3) The unification of mathematics study by eliminating separate hours for the study of arithmetic, algebra, and geometry, and using these subjects indiscriminately to help each other, especially in early years of the secondary school; (4) The early introduction of algebra through the generalization of the laws of arithmetic; (5) The use of the mathematics books of the school library as resource and enrichment material as well as a necessary part of mathematics study; (6) An ever increasing use and construction of gadgets and models; (7) Stressing the metric system by making it the only one to be learned and used; (8) Introducing some concepts of modern mathematics so as to prepare for the study of Modern Algebra at the University level, and also to give clarity and unity to the elementary mathematics and (9) introducing aspects of descriptive statistics in both arithmetic and algebra.

Elementary Mathematics in the U.S.A.

The present program in mathematics for youth age 6 years to 15 years in the U.S.A. can best be described as average when compared to that in other countries.[4] Up to about 1925, the teaching was for the most part a rote-manipulative-tool developing type of instruction. From 1925 to 1940, a social utility program was introduced whereby arithmetic and geometry were to be learned when and as they arose in social problems in the life of the youngster. This method of teaching proved a complete failure. Beginning around 1940, a meaning theory of teaching was advocated and is rapidly gaining favor. This theory stresses the teaching of arithmetic as an ordered structure of number, and geometry as a systemic structure of space, which is abstracted from physical objects and the world around us through the study of their characteristics and properties. In Arithmetic all learning is rationalized; the concepts of numerosity, order, numeration, and the fundamental operations are developed intuitively and used before any attempt is made to systematize the fundamental relations and facts.

Once the concept of counting is developed, the children discover, write, then memorize through drill, all the fundamental facts needed for the four fundamental operations with numbers. The commutative, associative, and distributive laws are used in learning the facts and the algorisms of the fundamental computational processes. Since subtraction is conceived fundamentally as removing a known subset from a given universe, and then the problem is to find the complement of the subset, we teach the take-away exchange algorism and not the Austrian Method. We have found, that for rationalization, this method pays real dividends, and we reserve the Austrian Method for the time when we introduce the new system of positive and negative numbers. Long division is used to introduce division with whole numbers using one, two or three digit divisors, and

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[4]) *The Development of Arithmetic as an Elementary School subject since 1900.* Sydney Tompkins. Unpublished Doctoral Study — Teachers College, Columbia University. 1957.

short division is reserved for the 7th and 8th school years when simplification and rationalization can be made. In general, we teach no mathematical operation that is beyond a genuine understanding and rationalization by the pupils, and we teach this mathematics by building a structure, based on the laws of number, and by abstracting from physical models. In the later years, grades 8 and 9, we produce models in which the ordinary arithmetic and geometry do not hold up, so that the pupils will understand there is other mathematics than that which they have learned, and that this other mathematics is useful.

The geometry of the 5th to 9th grades is entirely informal and intuitive and covers the ordinary study of shape, size, and position. The one really significant change that is occurring is in the 9th school year (age 15 years). Here the entire year will be devoted to developing elementary algebra from a modern point of view. The concepts, language, and symbolism of sets (Mengen, Ensembles) is introduced at the very start of the study. A variable is conceived as a symbol which may be replaced by any element (number) of its domain. We stress that we must always know the domain in which we are working. We talk of expressions, set-builders, and also refer to the roots of equations as the solution set. We are convinced that through this approach, using the five fundamental laws for a *Ring* (but not the word Ring), the algebra will achieve unity, clarity, meaning, and challenge to the intellect, that it never had before. Our approach to *function* from the start will be a mapping exhibited by a set of ordered pairs of numbers, and defined by a relation, that makes it single-valued. With these ideas we can introduce elementary methods of proof in algebra comparable to those heretofore reserved for geometry alone. We believe our experiment is well worth watching by all countries.

Conclusion.

Previously, it has been said, that all reports indicate a trend towards teaching for meaning. But meaning has different connotations to the different reporters. This report closes on a note of the necessity, because of our world culture today, of teaching all our mathematics, from the very first formal lessons, through the use of thinking. This does not mean the teaching of a rigorous mathematical structure, but that all learning should be through the use of cognitive intelligence, and not other mental abilities.

More and more the only services which human beings can offer to society and which society will need will be intellectual. The age of automation which appears immediately in front of us, and the expanding application of mathematics to other areas of knowledge than Physics, will demand from society not only greater but more brain power. Hence it depends upon the schools of the world to develop intellectual power, not only that which occurs in the top notch brains, but the power of each and every individual to the capacity that he has this power. The power to think to solve problems, to apply knowledge to practical situations, and to create, is present in all normal persons, though of course not to the same extent. Mathematics, both as liberal education for all pupils, and as special education for career people, offers a type of cognitive intellectual development so necessary for modern culture. But mathematics instruction will achieve its goal only if the teaching builds a structure of knowledge through the organization of concepts and relations of number and space and facility in the use of this structure, so as to give the learner a genuine insight into what mathematics is like in the twentieth century, and what it does.

Will the youth learn mathematics in this manner ? They will, and further, many who now desert their study of mathematics at their first opportunity, will not do so under cognitive learning. They will learn, and continue their study, because children are first of all motivated by intellectual curiosity and not by use or monetary values. They will continue their study because all of us like to do that which we understand, that which challenges us, and that which offers us a reasonable chance of success in the outcome.