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This result implies that

$$r_8(n) = 16(-1)^n \zeta_3(n) = 16(-1)^{n-1}\left(\sigma_3^0(n) - \sigma_3^e(n)\right), \tag{35}$$

where $\sigma_3^0(n)$ denotes the sum of the third powers of the odd divisors of n, and $\sigma_3^e(n)$ denotes the sum of the third powers of the even divisors of n. This is the desired result. [8]

5. Modular Transforms.

It has been shown in [2] that for $k > 1$, the function $\alpha_{2k-1}(t)$ satisfies the modular transformation

$$t^k \alpha_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \alpha_{2k-1}(2\pi/t). \tag{36}$$

For $k = 1$, the conditional convergence of the double series in (1) creates difficulties [9], which however, have been resolved by Hurwitz [3], who gives a result equivalent, in our notation, to the formula

$$t\,\alpha_1(2\pi t) = -\frac{1}{t} a_1(2\pi/t) + \frac{6}{\pi}. \tag{37}$$

We find that this result may be proved very easily by using (36) in conjunction with the relation

$$\alpha_5(t) = \alpha_3'(t) + \alpha_1(t)\,\alpha_3(t), \tag{38}$$

which is the case $n = 2$ in (26).

With the aid of equations (30), (31) and (33), the transforms (36) and (37) yield those for our functions $(4)_1$, $(5)_1$ and $(6)_1$. It is found that under the modular transformation in question, the first two functions are reciprocal in the sense that,

$$t^k \Psi_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \chi_{2k-1}(2\pi/t), \quad k \geq 1. \tag{39}$$

The remaining function (6), transforms in a manner analogous to $\alpha_{2k-1}(t)$, namely

$$t^k \Phi_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \Phi_{2k-1}(2\pi/t), \quad k > 1, \tag{40}$$

while for $k = 1$, the following holds:

$$t\,\Phi_1(2\pi t) = -\frac{1}{t}\,\Phi_1(2\pi t) - \frac{1}{4\pi}\,. \tag{41}$$

Finally, we note that for $t = 1$, (37) and (41) yield rapidly convergent series which are of interest, namely,

$$8\sum_{n=1}^{\infty} e^{-2\pi n}\,\sigma_1(n) = 1/3 - 1/\pi\,, \tag{42}$$

$$8\sum_{n=1}^{\infty} e^{-\pi n}\,\zeta_1(n) = -1/\pi\,. \tag{43}$$

These, in combination, give finally,

$$8\sum_{n=1}^{\infty} e^{-\pi n}\,\sigma_1^0(n) = 2/3 - 1/\pi \tag{44}$$

where $\sigma_1^0(n)$ is the sum of the *odd* divisors of n.

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[3] S. Ramanujan, *Collected Papers* (1927), p. 141 or *Trans. Camb. Phil. Soc.*, 22 (1916), pp. 159-184, Table I.

[4] Tannery-Molk, *Fonctions elliptiques*, t. 2, table XXXII, p. 252.

[5] See E. T. Bell, *Algebraic Arithmetic*; Colloquium Publications, No. 7, *Am. Math. Soc.*, pp. 146- (1927). Also, Carathéodory, *Funktionentheorie*, Bd. 1, p. 265 (1950).

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[8] Glaisher, J. W. L., *Proc. London Math. Soc.*, Series 2, 5 (1907), p. 479; G. H. Hardy and E. M. Wright, *Introd. Theory of Numbers*, 3rd ed. (1954), p. 314.

[9] In this connection, Professor van der Pol has very kindly brought to my attention the following references: (i) Riemann, *Ges. Werke*, p. 466, « Erläuterung von R. Dedekind » and (ii) Siegel, C. L., *Mathematika* (Univ. College, London), vol. 1, p. 4 (1954).

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