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Autor:	Basoco, M. A.
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This result implies that

$$(35) \quad r_8(n) = 16(-1)^n \zeta_3(n) = 16(-1)^{n-1} (\sigma_3^0(n) - \sigma_3^e(n)) ,$$

where $\sigma_3^0(n)$ denotes the sum of the third powers of the odd divisors of n , and $\sigma_3^e(n)$ denotes the sum of the third powers of the even divisors of n . This is the desired result. [8]

5. MODULAR TRANSFORMS.

It has been shown in [2] that for $k > 1$, the function $\alpha_{2k-1}(t)$ satisfies the modular transformation

$$(36) \quad t^k \alpha_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \alpha_{2k-1}(2\pi/t) .$$

For $k = 1$, the conditional convergence of the double series in (1) creates difficulties [9], which however, have been resolved by HURWITZ [3], who gives a result equivalent, in our notation, to the formula

$$(37) \quad t \alpha_1(2\pi t) = -\frac{1}{t} a_1(2\pi/t) + \frac{6}{\pi} .$$

We find that this result may be proved very easily by using (36) in conjunction with the relation

$$(38) \quad \alpha_5(t) = \alpha'_3(t) + \alpha_1(t) \alpha_3(t) ,$$

which is the case $n = 2$ in (26).

With the aid of equations (30), (31) and (33), the transforms (36) and (37) yield those for our functions (4)₁, (5)₁ and (6)₁. It is found that under the modular transformation in question, the first two functions are reciprocal in the sense that,

$$(39) \quad t^k \Psi_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \chi_{2k-1}(2\pi/t) , \quad k \geq 1 .$$

The remaining function (6), transforms in a manner analogous to $\alpha_{2k-1}(t)$, namely

$$(40) \quad t^k \Phi_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \Phi_{2k-1}(2\pi/t) , \quad k > 1 ,$$

while for $k = 1$, the following holds:

$$(41) \quad t \Phi_1(2\pi t) = -\frac{1}{t} \Phi_1(2\pi t) - \frac{1}{4\pi}.$$

Finally, we note that for $t = 1$, (37) and (41) yield rapidly convergent series which are of interest, namely,

$$(42) \quad 8 \sum_{n=1}^{\infty} e^{-2\pi n} \sigma_1(n) = \frac{1}{3} - \frac{1}{\pi},$$

$$(43) \quad 8 \sum_{n=1}^{\infty} e^{-\pi n} \zeta_1(n) = -\frac{1}{\pi}.$$

These, in combination, give finally,

$$(44) \quad 8 \sum_{n=1}^{\infty} e^{-\pi n} \sigma_1^0(n) = \frac{2}{3} - \frac{1}{\pi}$$

where $\sigma_1^0(n)$ is the sum of the *odd* divisors of n .

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- [4] TANNERY-MOLK, *Fonctions elliptiques*, t. 2, table XXXII, p. 252.
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*The University of Nebraska,
Lincoln 8, Nebraska
U.S.A.*