

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 4 (1958)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON CERTAIN ARITHMETICAL FUNCTIONS RELATED TO A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION
Autor: Basoco, M. A.
Kapitel: 4. The Functions $\Psi_{2k-1}(t)$, $X_{2k-1}(t)$, $\Phi_{2k-1}(t)$ as Double Sums.
DOI: <https://doi.org/10.5169/seals-34625>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

which is equivalent to a known recurrence for the BERNOULLI numbers [8].

4. THE FUNCTIONS $\Psi_{2k-1}(t)$, $X_{2k-1}(t)$, $\Phi_{2k-1}(t)$ AS DOUBLE SUMS.

The results which are stated as (4), (5), (6) follow readily from (1) and (2) which are known to be equivalent (see [1], [2]). It is to be observed first that a comparison of (4) and (5) with (1) taking into account (27) gives the relations:

$$(30) \quad \Psi_{2k-1}(t) = h_{2k-1}(t/2) - h_{2k-1}(t) = V_k(\alpha_{2k-1}(t/2) - \alpha_{2k-1}(t)),$$

$$(31) \quad X_{2k-1}(t) = 2^{2k} h_{2k-1}(2t) - h_{2k-1}(t) = V_k(2^{2k} \alpha_{2k-1}(2t) - \alpha_{2k-1}(t)).$$

From (4) and (6) we also have,

$$(32) \quad \Phi_{2k-1}(t) = 2^{2k} \Psi_{2k-1}(2t) - \Psi_{2k-1}(t).$$

By (30), we may write

$$(33) \quad \Phi_{2k-1}(t) = -V_k(\alpha_{2k-1}(t/2) - (2^{2k} + 1)\alpha_{2k-1}(t) + 2^{2k}\alpha_{2k-1}(2t)).$$

Thus, our functions (4), (5), (6) are expressed in terms of $\alpha_{2k-1}(u)$. These relations in conjunction with (1) and (2) identify them with $(4)_1$, $(5)_1$, and $(6)_1$ respectively.

It is of interest to note that (31) with $k = 2$ permits, with the aid of a result of VAN DER POL [1], the deduction of Jacobi's famous theorem on the number of representations $r_8(n)$ of the integer n as the sum of eight squares. Thus,

$$(34) \quad 240 X_3(t) = 16 \alpha_3(2t) - \alpha_3(t) = 15 \theta_0^8(0, q)$$

where $q = \exp(-t)$. Hence,

$$\theta_0^8(0, q) = 16 X_3(t) = 1 + 16 \sum_{n=1}^{\infty} q^n \zeta_3(n),$$

and

$$\theta_3^8(0, q) = 1 + 16 \sum_{n=1}^{\infty} (-1)^n q^n \zeta_3(n).$$

This result implies that

$$(35) \quad r_3(n) = 16 (-1)^n \zeta_3(n) = 16 (-1)^{n-1} (\sigma_3^0(n) - \sigma_3^e(n)) ,$$

where $\sigma_3^0(n)$ denotes the sum of the third powers of the odd divisors of n , and $\sigma_3^e(n)$ denotes the sum of the third powers of the even divisors of n . This is the desired result. [8]

5. MODULAR TRANSFORMS.

It has been shown in [2] that for $k > 1$, the function $\alpha_{2k-1}(t)$ satisfies the modular transformation

$$(36) \quad t^k \alpha_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \alpha_{2k-1}(2\pi/t) .$$

For $k = 1$, the conditional convergence of the double series in (1) creates difficulties [9], which however, have been resolved by HURWITZ [3], who gives a result equivalent, in our notation, to the formula

$$(37) \quad t \alpha_1(2\pi t) = -\frac{1}{t} \alpha_1(2\pi/t) + \frac{6}{\pi} .$$

We find that this result may be proved very easily by using (36) in conjunction with the relation

$$(38) \quad \alpha_5(t) = \alpha_3'(t) + \alpha_1(t) \alpha_3(t) ,$$

which is the case $n = 2$ in (26).

With the aid of equations (30), (31) and (33), the transforms (36) and (37) yield those for our functions $(4)_1$, $(5)_1$ and $(6)_1$. It is found that under the modular transformation in question, the first two functions are reciprocal in the sense that,

$$(39) \quad t^k \Psi_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \chi_{2k-1}(2\pi/t) , \quad k \geq 1 .$$

The remaining function (6), transforms in a manner analogous to $\alpha_{2k-1}(t)$, namely

$$(40) \quad t^k \Phi_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \Phi_{2k-1}(2\pi/t) , \quad k > 1 ,$$