Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 4 (1958)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON CERTAIN ARITHMETICAL FUNCTIONS RELATED TO A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION
Autor: Basoco, M. A.
Kapitel: 3. Recurrences.
DOI: https://doi.org/10.5169/seals-34625

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 27.03.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

powers of Ψ are lowered into subscripts; thus $\Psi^{(1)}$ would then be written $\Psi_1(t)$.

If (18) and (19) are substituted in (13), there results the following umbral identity:

$$(20) \qquad \Psi(1 - \cos \Psi s) + 2\frac{\partial}{\partial t}\left(\frac{1 - \cos \Psi s}{\Psi}\right) = 2 \sin \Psi s * \sin \Psi s ,$$

where the asterisk (*) indicates umbral multiplication.

For the cases $r = 2, 3$, rather extensive calculations show that umbral identities of the same form exist. We may therefore state the following result which is implied by the non-linear equation (13).

Theorem 1: "Let Ψ, X, Φ be respectively the umbrae of the sequences $\{\Psi_{2k-1}(t)\}$, $\{X_{2k-1}(t)\}$, and $\{\Phi_{2k-1}(t)\}$. If γ is one of these umbrae, then the following umbral identity holds:

$$(21) \qquad \gamma(1 - \cos \gamma s) + 2\frac{\partial}{\partial t}\left(\frac{1 - \cos \gamma s}{\gamma}\right) = 2 \sin \gamma s * \sin \gamma s .$$

3. Recurrences.

It is clear that (20) implies a recurrence relation for the functions $\Psi_j(t)$, and indeed, Theorem 1 yields the following.

Theorem 2: "Let $\gamma_j(t)$ be $\Psi_j(t)$, $X_j(t)$ or $\Phi_j(t)$; then the following recurrence holds:

$$(22) \qquad \frac{d}{dt}\gamma_{2n-1}(t) + \frac{1}{2}\gamma_{2n+1}(t) = \sum_{k=0}^{n-1}\binom{2n}{2k+1}\gamma_{2k+1}(t)\,\gamma_{2n-2k-1}(t) ,$$

and hence $\gamma_{2n+1}(t)$ is a polynomial in $\gamma_1(t)$ and its derivatives up to order n."

This result, in turn, implies the following

Theorem 3: "Let $\rho_{2k-1}(n)$ be either of the arithmetical functions $\beta_{2k-1}(n)$ or $\xi_{2k-1}(n)$ defined by (8) and (10) respectively; then $\rho_{2k-1}(n)$ satisfies a recurrence relation of the form:

$$(23) \qquad \rho_{2k+1}(n) - n\,\rho_{2k-1}(n) = 2\sum_{s=0}^{k-1}\sum_{j=1}^{n-1}\binom{2k}{2s+1}\rho_{2s+1}(j)\,\rho_{2k-2s-1}(n-j),$$

for all n and $k \geq 1$. Moreover, the arithmetical function $\zeta_{2k-1}(n)$ defined by (9) satisfies the recurrence

$$(24)\quad \zeta_{2k+1}(n) - 2n\,\zeta_{2k-1}(n) = 2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\Big\{U_{k-s}\,\zeta_{2s+1}(n) + U_{s+1}\,\zeta_{2k-2s-1}(n) + \sum_{j=1}^{n-1}\zeta_{2s+1}(j)\,\zeta_{2k-2s-1}(n-j)\Big\}$$

where U_k is defined by (11) and $n, k \geq 1$."

Incidentally, the comparison of coefficients which yields (24) also gives:

$$(25)\quad U_{n+1} = 2\sum_{k=0}^{n-1}\binom{2n}{2k+1}U_{k+1}\,U_{n-k}\,, \quad n \geq 1\,,$$

which is equivalent to a result given by NIELSEN [7].

Finally, the case $r = 1$, has been discussed by VAN DER POL [1] who finds an expression analogous to (22) as follows:

$$(26)\quad \frac{d}{dt}h_{2n-1}(t) + \frac{2n+3}{4n+2}h_{2n+1}(t) = \sum_{k=0}^{n-1}\binom{2n}{2k+1}h_{2k+1}(t)\,h_{2n-2k-1}(t)\,, \quad n \geq 1\,,$$

where,

$$(27)\quad h_{2n-1}(t) = \frac{(-1)^n B_n}{4n}\alpha_{2n-1}(t)\,,$$

$\alpha_{2n-1}(t)$ being defined by (2).

We find the analogue of (24) for this case to be:

$$(28)\quad \frac{2k+3}{2k+1}\sigma_{2k+1}(n) - 2n\,\sigma_{2k-1}(n) = 2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\Big\{V_{k-s}\,\sigma_{2s+1}(n) + V_{s+1}\,\sigma_{2k-2s-1}(n) + \sum_{j=1}^{n-1}\sigma_{2s+1}(j)\,\sigma_{2k-2s-1}(n-j)\Big\}\,, \quad n \geq 1\,.$$

Corresponding to (25), we find

$$(29)\quad V_{n+1} = \frac{4n+2}{2n+3}\sum_{k=0}^{n-1}\binom{2n}{2k+1}V_{k+1}\,V_{n-k}\,,$$

which is equivalent to a known recurrence for the BERNOULLI numbers [8].

4. THE FUNCTIONS $\Psi_{2k-1}(t)$, $X_{2k-1}(t)$, $\Phi_{2k-1}(t)$ AS DOUBLE SUMS.

The results which are stated as (4), (5), (6) follow readily from (1) and (2) which are known to be equivalent (see [1], [2]). It is to be observed first that a comparison of (4) and (5) with (1) taking into account (27) gives the relations:

$$\Psi_{2k-1}(t) = h_{2k-1}(t/2) - h_{2k-1}(t) = V_k\left(\alpha_{2k-1}(t/2) - \alpha_{2k-1}(t)\right), \tag{30}$$

$$X_{2k-1}(t) = 2^{2k} h_{2k-1}(2t) - h_{2k-1}(t) = V_k\left(2^{2k}\alpha_{2k-1}(2t) - \alpha_{2k-1}(t)\right). \tag{31}$$

From (4) and (6) we also have,

$$\Phi_{2k-1}(t) = 2^{2k}\Psi_{2k-1}(2t) - \Psi_{2k-1}(t). \tag{32}$$

By (30), we may write

$$\Phi_{2k-1}(t) = -V_k\left(\alpha_{2k-1}(t/2) - (2^{2k}+1)\alpha_{2k-1}(t) + 2^{2k}\alpha_{2k-1}(2t)\right). \tag{33}$$

Thus, our functions (4), (5), (6) are expressed in terms of $\alpha_{2k-1}(u)$. These relations in conjunction with (1) and (2) identify them with $(4)_1$, $(5)_1$, and $(6)_1$ respectively.

It is of interest to note that (31) with $k = 2$ permits, with the aid of a result of VAN DER POL [1], the deduction of Jacobi's famous theorem on the number of representations $r_8(n)$ of the integer n as the sum of eight squares. Thus,

$$240\, X_3(t) = 16\,\alpha_3(2t) - \alpha_3(t) = 15\,\theta_0^8(0, q) \tag{34}$$

where $q = \exp(-t)$. Hence,

$$\theta_0^8(0, q) = 16\, X_3(t) = 1 + 16\sum_{n=1}^{\infty} q^n\,\zeta_3(n),$$

and

$$\theta_3^8(0, q) = 1 + 16\sum_{n=1}^{\infty}(-1)^n q^n\,\zeta_3(n).$$