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$$(11) \quad U_k = (2^{2k} - 1) V_k = (-1)^k (2^{2k} - 1) \frac{B_k}{4k}.$$

As is well known, the double series occurring in (1), (4), (5), (6) are absolutely convergent for  $k > 1$ ; for  $k = 1$ , the convergence is conditional. However, as has been shown by HURWITZ [2] in the case of (1), if the summation is first carried out with respect to  $m$  and then with respect to  $n$ , the resulting sum agrees with (2) with  $k = 1$ . For this case ( $k = 1$ ) similar conditions hold for (4), (5) and (6). These matters are of relevance in studying certain modular transformations of these functions to be discussed later.

## 2. UMBRAL RELATIONS.

The functions defined in what precedes arise in a natural manner as a consequence of the well-known fact that the Jacobi theta functions are solutions of the partial differential equation

$$(12) \quad \frac{\partial^2 z}{\partial s^2} = 2 \frac{\partial z}{\partial t}, \quad z = \theta_r(\nu, \tau), \quad (r = 1, 2, 3, 4),$$

with  $s = 2\pi\nu$  and  $-t = 2\pi i\tau$ , and, what appears to be less well-known, that the functions  $u = \ln\theta_r(\nu, \tau)$  satisfy the non-linear equation:

$$(13) \quad \frac{\partial^2 u}{\partial s^2} = 2 \frac{\partial u}{\partial t} - \left(\frac{\partial u}{\partial s}\right)^2.$$

Here, the notation for the theta function is that used in TANNERY-MOLK's treatise [4].

The arithmetical consequence of (13) can best be obtained through the use of the infinite product representation of  $\theta_r(\nu, \tau)$ . It is found that the calculations needed are greatly facilitated and the results obtained very simply expressed in a symbolic form through an application of the umbral calculus of BLISSARD and LUCAS [5]. It is not feasible to give details for all cases and we merely indicate briefly the nature of the calculations for the case  $r = 4$ . Thus, since,

$$(14) \quad \theta_4(\nu, \tau) = Q_0 \prod_{n=1}^{\infty} (1 - q^{2n-1} e^{2\pi i \nu}) (1 - q^{2n-1} e^{-2\pi i \nu}),$$

$$Q_0 = \prod_{n=1}^{\infty} (1 - q^{2n}), \quad q = e^{\pi i \tau},$$

and taking into account the change in variables from  $(\nu, \tau)$  to  $(s, t)$  it is found that if  $u(s, t) = \ln \theta_4(\nu, \tau)$ , then

$$(15) \quad \frac{\partial u}{\partial s} = 2 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \sin ns, \quad q = e^{-t/2},$$

and

$$(16) \quad \frac{\partial^2 u}{\partial s^2} = 2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos ns;$$

moreover,

$$(17) \quad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} - 2 \frac{\partial}{\partial t} \left\{ \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \frac{\cos ns}{n} \right\}$$

provided  $\operatorname{Re} t \pm 2\operatorname{Im} s > 0$  in (15), (16) and (17).

Now, in (15) replace  $\sin ns$  by its power series development and interchange the order of summation to obtain

$$(15)_1 \quad \frac{\partial u}{\partial s} = 2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{s^{2k-1}}{(2k-1)!} \Psi_{2k-1}(t).$$

Hence if  $\Psi$  is the umbra of the sequence  $\{\Psi_{2k-1}(t)\}$  we may write symbolically:

$$(18) \quad \frac{\partial u}{\partial s} \cong 2 \sin \Psi s, \quad \frac{\partial^2 u}{\partial s^2} \cong 2 \Psi \cos \Psi s.$$

Similarly, a more extended calculation shows that

$$(19) \quad \frac{\partial u}{\partial t} \cong \Psi^{(1)} + 2 \frac{\partial}{\partial t} \frac{1 - \cos \Psi s}{\Psi}.$$

In (18) and (19), in order to pass from symbolic equality to actual equality, the functions  $\sin \Psi s$ ,  $\cos \Psi s$  and  $(1 - \cos \Psi s)/\Psi$  are to be expanded in powers of  $s$  and then the exponents in the

powers of  $\Psi$  are lowered into subscripts; thus  $\Psi^{(1)}$  would then be written  $\Psi_1(t)$ .

If (18) and (19) are substituted in (13), there results the following umbral identity:

$$(20) \quad \Psi (1 - \cos \Psi_s) + 2 \frac{\partial}{\partial t} \left( \frac{1 - \cos \Psi_s}{\Psi} \right) = 2 \sin \Psi_s * \sin \Psi_s ,$$

where the asterisk (\*) indicates umbral multiplication.

For the cases  $r = 2, 3$ , rather extensive calculations show that umbral identities of the same form exist. We may therefore state the following result which is implied by the non-linear equation (13).

*Theorem 1:* "Let  $\Psi$ ,  $X$ ,  $\Phi$  be respectively the umbrae of the sequences  $\{\Psi_{2k-1}(t)\}$ ,  $\{X_{2k-1}(t)\}$ , and  $\{\Phi_{2k-1}(t)\}$ . If  $\gamma$  is one of these umbrae, then the following umbral identity holds:

$$(21) \quad \gamma (1 - \cos \gamma_s) + 2 \frac{\partial}{\partial t} \left( \frac{1 - \cos \gamma_s}{\gamma} \right) = 2 \sin \gamma_s * \sin \gamma_s .$$

### 3. RECURRENCES.

It is clear that (20) implies a recurrence relation for the functions  $\Psi_j(t)$ , and indeed, Theorem 1 yields the following.

*Theorem 2:* "Let  $\gamma_j(t)$  be  $\Psi_j(t)$ ,  $X_j(t)$  or  $\Phi_j(t)$ ; then the following recurrence holds:

$$(22) \quad \frac{d}{dt} \gamma_{2n-1}(t) + \frac{1}{2} \gamma_{2n+1}(t) = \sum_{k=0}^{n-1} \binom{2n}{2k+1} \gamma_{2k+1}(t) \gamma_{2n-2k-1}(t) ,$$

and hence  $\gamma_{2n+1}(t)$  is a polynomial in  $\gamma_1(t)$  and its derivatives up to order  $n$ ."

This result, in turn, implies the following

*Theorem 3:* "Let  $\rho_{2k-1}(n)$  be either of the arithmetical functions  $\beta_{2k-1}(n)$  or  $\xi_{2k-1}(n)$  defined by (8) and (10) respectively; then  $\rho_{2k-1}(n)$  satisfies a recurrence relation of the form:

$$(23) \quad \rho_{2k+1}(n) - n \rho_{2k-1}(n) = 2 \sum_{s=0}^{k-1} \sum_{j=1}^{n-1} \binom{2k}{2s+1} \rho_{2s+1}(j) \rho_{2k-2s-1}(n-j) ,$$