Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 4 (1958)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON CERTAIN ARITHMETICAL FUNCTIONS RELATED TO A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION

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DOI: https://doi.org/10.5169/seals-34625

ON CERTAIN ARITHMETICAL FUNCTIONS RELATED TO A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION [1)]

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(Reçu le 21 mars 1957)

1. INTRODUCTION.

In a recent paper, VAN DER POL [1] has made an extensive study of the elliptic modular functions defined by:

$$(1) \qquad \alpha_{2k-1}(t) = \frac{\sum\limits_{m=-\infty}^{\infty}{}' \sum\limits_{n=-\infty}^{\infty} (m + n\tau)^{-2k}}{\sum\limits_{m=-\infty}^{\infty}{}' m^{-2k}}, \quad k = 2, 3, 4, 5, \ldots$$

where $-t = 2\pi i\tau$, $\mathrm{Im}\tau > 0$; m, n range over all integral values and $(m, n) \neq (0, 0)$. HURWITZ [2] and VAN DER POL [1] have shown by different methods that these functions have series representations of the form

$$(2) \quad \alpha_{2k-1}(t) = 1 + \frac{4(-1)^k_k}{B_k} \sum_{n=1}^{\infty} \frac{n^{2k-1} e^{-nt}}{1 - e^{-nt}} = 1 + \frac{4k(-1)^k}{B_k} \sum_{n=1}^{\infty} e^{-nt} \sigma_{2k-1}(n),$$

where $\sigma_{2k-1}(n)$ is the sum of the $(2k-1)$-*st* powers of the integral divisors of n, and B_k are the Bernoulli numbers. These functions are closely related to the coefficients in the series development of the Weierstrass function $\wp(u)$, and may be found tabulated in VAN DER POL's paper [1] as well as in a paper by RAMANUJAN [3] who uses a different notation.

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1) This paper was prepared while the author held a Faculty Research Fellowship during the summer of 1956, granted by the University of Nebraska Research Council.

The integers (m, n) in (1) are unrestricted as to parity, and it is of some interest to consider the three sequences of functions for which the corresponding defining double sum is restricted by the conditions that

$$(\mathrm{m}, \mathrm{n}) \equiv (0,1),\ (1,0) \text{ or } (1,1), \bmod 2.$$

These functions are defined respectively, for $k > 1$, by

$$(4)\quad \Psi_{2k-1}(t) = C_k \sum_{(\mu)} \sum_{(\nu)} (\mu + \nu\tau)^{-2k}\,,$$

$$(5)\quad X_{2k-1}(t) = C_k \sum_{(\mu)} \sum_{(\nu)} (\nu + \mu\tau)^{-2k}\,, \qquad \begin{pmatrix} \mu = 0 \pm 2, \pm 4, \ldots \\ \nu = \pm 1, + 3, \pm 5, \ldots \end{pmatrix}$$

$$(6)\quad \Phi_{2k-1}(t) = C_k \sum_{(\rho)} \sum_{(\sigma)} (\rho + \sigma\tau)^{-2k}\,, \qquad (\rho, \sigma = \pm 1, \pm 3, \pm 5, \ldots).$$

where,

$$(7)\quad C_k \sum_{(\mu)}{}' \mu^{-2k} = V_k\,, \quad V_k = \frac{(-1)^k B_k}{4k}\,; \quad C_k = \frac{(-1)^k (2k-1)!}{2^{2k+1}\pi^{2k}}\,.$$

Written in arithmetical form these functions will be shown to have the form:

$$(4)_1\quad \Psi_{2k-1}(t) = \sum_{n=1}^{\infty} \frac{n^{2k-1} e^{-nt/2}}{1 - e^{-nt}} = \sum_{n=1}^{\infty} e^{-nt/2} \beta_{2k-1}(n)\,,$$

$$(5)_1\quad X_{2k-1}(t) = U_k + \sum_{n=1}^{\infty} \frac{(-1)^n n^{2k-1} e^{-nt}}{1 - e^{-nt}} = U_k + \sum_{n=1}^{\infty} e^{-nt} \zeta_{2k-1}(n)\,,$$

$$(6)_1\quad \Phi_{2k-1}(t) = \sum_{n=1}^{\infty} \frac{(-1)^k n^{2k-1} e^{-nt/2}}{1 - e^{-nt}} = e^{-nt/2} \xi_{2k-1}(n)\,,$$

where,

(8) $\beta_{2k-1}(n)$ = sum of the $(2k-1)$ *st* powers of the integral divisors of n whose conjugates are odd.

(9) $\zeta_{2k-1}(n)$ = (sum of the $(2k-1)$ *st* powers of the even divisors of n) — (sum of the $(2k-1)$ — *st* powers of the odd divisors of n).

(10) $\xi_{2k-1}(n)$ = (sum of the $(2k-1)$ *st* powers of the even divisors of n whose conjugates are odd) — (sum of the $(2k-1)$ — *st* powers of the odd divisors of n whose conjugates are odd);

$$U_k = (2^{2k} - 1)\, V_k = (-1)^k (2^{2k} - 1) \frac{B_k}{4k}. \tag{11}$$

As is well known, the double series occurring in (1), (4), (5), (6) are absolutely convergent for $k > 1$; for $k = 1$, the convergence is conditional. However, as has been shown by HURWITZ [2] in the case of (1), if the summation is first carried out with respect to m and then with respect to n, the resulting sum agrees with (2) with $k = 1$. For this case ($k = 1$) similar conditions hold for (4), (5) and (6). These matters are of relevance in studying certain modular transformations of these functions to be discussed later.

2. UMBRAL RELATIONS.

The functions defined in what precedes arise in a natural manner as a consequence of the well-known fact that the Jacobi theta functions are solutions of the partial differential equation

$$\frac{\partial^2 z}{\partial s^2} = 2 \frac{\partial z}{\partial t}, \quad z = \theta_r(v, \tau), \quad (r = 1, 2, 3, 4), \tag{12}$$

with $s = 2\pi v$ and $-t = 2\pi i \tau$, and, what appears to be less well-known, that the functions $u = ln\theta_r(v, \tau)$ satisfy the non-linear equation:

$$\frac{\partial^2 u}{\partial s^2} = 2 \frac{\partial u}{\partial t} - \left(\frac{\partial u}{\partial s}\right)^2. \tag{13}$$

Here, the notation for the theta function is that used in TANNERY-MOLK's treatise [4].

The arithmetical consequence of (13) can best be obtained through the use of the infinite product representation of $\theta_r(v, \tau)$. It is found that the calculations needed are greatly facilitated and the results obtained very simply expressed in a symbolic form through an application of the umbral calculus of BLISSARD and LUCAS [5]. It is not feasible to give details for all cases and we merely indicate briefly the nature of the calculations for the case $r = 4$. Thus, since,

$$(14) \qquad \theta_4(v, \tau) = Q_0 \prod_{n=1}^{\infty} (1 - q^{2n-1} e^{2\pi i v})(1 - q^{2n-1} e^{-2\pi i v}) ,$$

$$Q_0 = \prod_{n=1}^{\infty} (1 - q^{2n}) , \quad q = e^{\pi i \tau} ,$$

and taking into account the change in variables from (v, τ) to (s, t) it is found that if $u(s, t) = ln\, \theta_4(v, \tau)$, then

$$(15) \qquad \frac{\partial u}{\partial s} = 2 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \sin ns , \quad q = e^{-t/_2} ,$$

and

$$(16) \qquad \frac{\partial^2 u}{\partial s^2} = 2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos ns ;$$

moreover,

$$(17) \qquad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} - 2 \frac{\partial}{\partial t} \left\{ \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \frac{\cos ns}{n} \right\}$$

provided $Re t \pm 2 Im\, s > 0$ in (15), (16) and (17).

Now, in (15) replace sin ns by its power series development and interchange the order of summation to obtain

$$(15)_1 \qquad \frac{\partial u}{\partial s} = 2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{s^{2k-1}}{(2k-1)!} \Psi_{2k-1}(t) .$$

Hence if Ψ is the umbra of the sequence $\{\Psi_{2k-1}(t)\}$ we may write symbolically:

$$(18) \qquad \frac{\partial u}{\partial s} \cong 2 \sin \Psi s , \qquad \frac{\partial^2 u}{\partial s^2} \cong 2\, \Psi \cos \Psi s .$$

Similarly, a more extended calculation shows that

$$(19) \qquad \frac{\partial u}{\partial t} \cong \Psi^{(1)} + 2 \frac{\partial}{\partial t} \frac{1 - \cos \Psi s}{\Psi} .$$

In (18) and (19), in order to pass from symbolic equality to actual equality, the functions $\sin \Psi s$, $\cos \Psi s$ and $(1 - \cos \Psi s)/\Psi$ are to be expanded in powers of s and then the exponents in the powers of Ψ are lowered into subscripts; thus $\Psi^{(1)}$ would then be written $\Psi_1(t)$.

If (18) and (19) are substituted in (13), there results the following umbral identity:

$$(20) \qquad \Psi(1 - \cos \Psi s) + 2 \frac{\partial}{\partial t}\left(\frac{1 - \cos \Psi s}{\Psi}\right) = 2 \sin \Psi s * \sin \Psi s \, ,$$

where the asterisk (*) indicates umbral multiplication.

For the cases $r = 2, 3$, rather extensive calculations show that umbral identities of the same form exist. We may therefore state the following result which is implied by the non-linear equation (13).

Theorem 1: "Let Ψ, X, Φ be respectively the umbrae of the sequences $\{\Psi_{2k-1}(t)\}$, $\{X_{2k-1}(t)\}$, and $\{\Phi_{2k-1}(t)\}$. If γ is one of these umbrae, then the following umbral identity holds:

$$(21) \qquad \gamma(1 - \cos \gamma s) + 2 \frac{\partial}{\partial t}\left(\frac{1 - \cos \gamma s}{\gamma}\right) = 2 \sin \gamma s * \sin \gamma s \, .$$

3. Recurrences.

It is clear that (20) implies a recurrence relation for the functions $\Psi_j(t)$, and indeed, Theorem 1 yields the following.

Theorem 2: "Let $\gamma_j(t)$ be $\Psi_j(t)$, $X_j(t)$ or $\Phi_j(t)$; then the following recurrence holds:

$$(22) \qquad \frac{d}{dt}\gamma_{2n-1}(t) + \frac{1}{2}\gamma_{2n+1}(t) = \sum_{k=0}^{n-1}\binom{2n}{2k+1}\gamma_{2k+1}(t)\,\gamma_{2n-2k-1}(t) \, ,$$

and hence $\gamma_{2n+1}(t)$ is a polynomial in $\gamma_1(t)$ and its derivatives up to order n."

This result, in turn, implies the following

Theorem 3: "Let $\rho_{2k-1}(n)$ be either of the arithmetical func tions $\beta_{2k-1}(n)$ or $\xi_{2k-1}(n)$ defined by (8) and (10) respectively; then $\rho_{2k-1}(n)$ satisfies a recurrence relation of the form:

$$(23) \qquad \rho_{2k+1}(n) - n\,\rho_{2k-1}(n) = 2\sum_{s=0}^{k-1}\sum_{j=1}^{n-1}\binom{2k}{2s+1}\rho_{2s+1}(j)\,\rho_{2k-2s-1}(n-j),$$

for all n and $k \geq 1$. Moreover, the arithmetical function $\zeta_{2k-1}(n)$ defined by (9) satisfies the recurrence

$$\begin{aligned}(24)\quad \zeta_{2k+1}(n) - 2n\,\zeta_{2k-1}(n) = 2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\Big\{ & \mathrm{U}_{k-s}\,\zeta_{2s+1}(n) + \\ & + \mathrm{U}_{s+1}\,\zeta_{2k-2s-1}(n) + \sum_{j=1}^{n-1}\zeta_{2s+1}(j)\,\zeta_{2k-2s-1}(n-j)\Big\}\end{aligned}$$

where U_k is defined by (11) and $n, k \geq 1$."

Incidentally, the comparison of coefficients which yields (24) also gives:

$$(25)\qquad \mathrm{U}_{n+1} = 2\sum_{k=0}^{n-1}\binom{2n}{2k+1}\mathrm{U}_{k+1}\,\mathrm{U}_{n-k}\,, \quad n \geq 1\,,$$

which is equivalent to a result given by NIELSEN [7].

Finally, the case $r = 1$, has been discussed by VAN DER POL [1] who finds an expression analogous to (22) as follows:

$$(26)\quad \frac{d}{dt}h_{2n-1}(t) + \frac{2n+3}{4n+2}h_{2n+1}(t) = \sum_{k=0}^{n-1}\binom{2n}{2k+1}h_{2k+1}(t)\,h_{2n-2k-1}(t)\,, \quad n \geq 1\,,$$

where,

$$(27)\qquad h_{2n-1}(t) = \frac{(-1)^n \mathrm{B}_n}{4n}\alpha_{2n-1}(t)\,,$$

$\alpha_{2n-1}(t)$ being defined by (2).

We find the analogue of (24) for this case to be:

$$\begin{aligned}(28)\quad \frac{2k+3}{2k+1}\sigma_{2k+1}(n) - 2n\,\sigma_{2k-1}(n) = 2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\Big\{ & \mathrm{V}_{k-s}\,\sigma_{2s+1}(n) + \\ & + \mathrm{V}_{s+1}\,\sigma_{2k-2s-1}(n) + \sum_{j=1}^{n-1}\sigma_{2s+1}(j)\,\sigma_{2k-2s-1}(n-j)\Big\}\,, \quad n \geq 1\,.\end{aligned}$$

Corresponding to (25), we find

$$(29)\qquad \mathrm{V}_{n+1} = \frac{4n+2}{2n+3}\sum_{k=0}^{n-1}\binom{2n}{2k+1}\mathrm{V}_{k+1}\,\mathrm{V}_{n-k}\,,$$

which is equivalent to a known recurrence for the BERNOULLI numbers [8].

4. THE FUNCTIONS $\Psi_{2k-1}(t)$, $X_{2k-1}(t)$, $\Phi_{2k-1}(t)$ AS DOUBLE SUMS.

The results which are stated as (4), (5), (6) follow readily from (1) and (2) which are known to be equivalent (see [1], [2]). It is to be observed first that a comparison of (4) and (5) with (1) taking into account (27) gives the relations:

$$\Psi_{2k-1}(t) = h_{2k-1}(t/2) - h_{2k-1}(t) = V_k\left(\alpha_{2k-1}(t/2) - \alpha_{2k-1}(t)\right), \tag{30}$$

$$X_{2k-1}(t) = 2^{2k} h_{2k-1}(2t) - h_{2k-1}(t) = V_k\left(2^{2k}\alpha_{2k-1}(2t) - \alpha_{2k-1}(t)\right). \tag{31}$$

From (4) and (6) we also have,

$$\Phi_{2k-1}(t) = 2^{2k}\Psi_{2k-1}(2t) - \Psi_{2k-1}(t). \tag{32}$$

By (30), we may write

$$\Phi_{2k-1}(t) = -V_k\left(\alpha_{2k-1}(t/2) - (2^{2k}+1)\alpha_{2k-1}(t) + 2^{2k}\alpha_{2k-1}(2t)\right). \tag{33}$$

Thus, our functions (4), (5), (6) are expressed in terms of $\alpha_{2k-1}(u)$. These relations in conjunction with (1) and (2) identify them with $(4)_1$, $(5)_1$, and $(6)_1$ respectively.

It is of interest to note that (31) with $k = 2$ permits, with the aid of a result of VAN DER POL [1], the deduction of Jacobi's famous theorem on the number of representations $r_8(n)$ of the integer n as the sum of eight squares. Thus,

$$240\, X_3(t) = 16\,\alpha_3(2t) - \alpha_3(t) = 15\,\theta_0^8(0, q) \tag{34}$$

where $q = \exp(-t)$. Hence,

$$\theta_0^8(0, q) = 16\, X_3(t) = 1 + 16\sum_{n=1}^{\infty} q^n \zeta_3(n),$$

and

$$\theta_3^8(0, q) = 1 + 16\sum_{n=1}^{\infty} (-1)^n q^n \zeta_3(n).$$

This result implies that

$$r_8(n) = 16(-1)^n \zeta_3(n) = 16(-1)^{n-1}\left(\sigma_3^0(n) - \sigma_3^e(n)\right) , \tag{35}$$

where $\sigma_3^0(n)$ denotes the sum of the third powers of the odd divisors of n, and $\sigma_3^e(n)$ denotes the sum of the third powers of the even divisors of n. This is the desired result. [8]

5. Modular Transforms.

It has been shown in [2] that for $k > 1$, the function $\alpha_{2k-1}(t)$ satisfies the modular transformation

$$t^k \alpha_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \alpha_{2k-1}(2\pi/t) . \tag{36}$$

For $k = 1$, the conditional convergence of the double series in (1) creates difficulties [9], which however, have been resolved by Hurwitz [3], who gives a result equivalent, in our notation, to the formula

$$t\,\alpha_1(2\pi t) = -\frac{1}{t} a_1(2\pi/t) + \frac{6}{\pi} . \tag{37}$$

We find that this result may be proved very easily by using (36) in conjunction with the relation

$$\alpha_5(t) = \alpha_3'(t) + \alpha_1(t)\,\alpha_3(t) , \tag{38}$$

which is the case $n = 2$ in (26).

With the aid of equations (30), (31) and (33), the transforms (36) and (37) yield those for our functions $(4)_1$, $(5)_1$ and $(6)_1$. It is found that under the modular transformation in question, the first two functions are reciprocal in the sense that,

$$t^k \Psi_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \chi_{2k-1}(2\pi/t) , \quad k \geq 1 . \tag{39}$$

The remaining function (6), transforms in a manner analogous to $\alpha_{2k-1}(t)$, namely

$$t^k \Phi_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k} \Phi_{2k-1}(2\pi/t) , \quad k > 1 , \tag{40}$$

while for $k = 1$, the following holds:

$$t\,\Phi_1(2\pi t) = -\frac{1}{t}\,\Phi_1(2\pi t) - \frac{1}{4\pi}\,. \tag{41}$$

Finally, we note that for $t = 1$, (37) and (41) yield rapidly convergent series which are of interest, namely,

$$8\sum_{n=1}^{\infty} e^{-2\pi n}\,\sigma_1(n) = \tfrac{1}{3} - 1/\pi\,, \tag{42}$$

$$8\sum_{n=1}^{\infty} e^{-\pi n}\,\zeta_1(n) = -\,1/\pi\,. \tag{43}$$

These, in combination, give finally,

$$8\sum_{n=1}^{\infty} e^{-\pi n}\,\sigma_1^0(n) = 2/3 - 1/\pi \tag{44}$$

where $\sigma_1^0(n)$ is the sum of the *odd* divisors of n.

REFERENCES

[1] Balth. VAN DER POL, *Koninkl. Nederl. Akad. van Wetenschappen (Amsterdam) Proceedings*, Ser. A, 54, No. 3 (1951).

[2] Adolf HURWITZ, *Math. Werke*, Bd. 1, pp. 19-26 and pp. 581-586; *Math. Annalen*, Bd. 18 (1881),p. 528; *ibid.*, Bd. 58 (1904), p. 343.

[3] S. RAMANUJAN, *Collected Papers* (1927), p. 141 or *Trans. Camb. Phil. Soc.*, 22 (1916), pp. 159-184, Table I.

[4] TANNERY-MOLK, *Fonctions elliptiques*, t. 2, table XXXII, p. 252.

[5] See E. T. BELL, *Algebraic Arithmetic*; Colloquium Publications, No. 7, *Am. Math. Soc.*, pp. 146- (1927). Also, CARATHÉODORY, *Funktionentheorie*, Bd. 1, p. 265 (1950).

[6] NIELSEN, N., *Traité élém. des nombres de Bernoulli*, pp. 45 and 56 (1923).

[7] NIELSEN, N., *ibid.*, p. 42 (13).

[8] GLAISHER, J. W. L., *Proc. London Math. Soc.*, Series 2, 5 (1907), p. 479; G. H. HARDY and E. M. WRIGHT, *Introd. Theory of Numbers*, 3rd ed. (1954), p. 314.

[9] In this connection, Professor van der Pol has very kindly brought to my attention the following references: (i) RIEMANN, *Ges. Werke*, p. 466, « Erläuterung von R. Dedekind » and (ii) SIEGEL, C. L., *Mathematika* (Univ. College, London), vol. 1, p. 4 (1954).

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