of the sequence we can speak of changes of sign only at the places $j \leq 2n - 2$.

THEOREM 3. *Let $c_{r,2n}$ denote the probability that there exist exactly r indices j such that*

$$S_{j-1} S_{j+1} < 0, \qquad 1 \leq j \leq 2n-1. \tag{5.1}$$

Then

$$c_{r,2n} = \frac{1}{2^{2n-2}} \binom{2n-1}{n-1-r}. \tag{5.2}$$

Proof. Let us say that two sequences $S_1, \ldots, S_m$ and $S'_1, \ldots, S'_m$ are *similar* if $|S_j| = |S'_j|$ for $j = 1, 2, \ldots, m$. Obviously $-S_1, -S_2, \ldots, -S_{2n}$ represents the only sequence similar to $S_1, \ldots, S_{2n}$ and such that changes of sign occur at the same places. On the other hand, if exactly k among the terms $S_1, \ldots, S_{2n-2}$ vanish, there exist exactly 2^{k+1} sequences similar to the sequence $S_1, \ldots, S_{2n}$. Out of k places we may choose r places in $\binom{k}{r}$ different ways, and it is therefore seen that

$$\begin{aligned} c_{r,2n} &= 2 \sum_{k=r}^{n-1} \binom{k}{r} 2^{-(k+1)} z_{k,2n-2} \\ &= \frac{1}{2^{2n-2}} \sum_{k=r}^{n-1} \binom{k}{r} \binom{2n-2-k}{n-1}. \end{aligned} \tag{5.3}$$

A well-known formula for binomial coefficients [3] which can be proved by induction now shows that (5.2) is true.

In (5.2) we recognize the binomial distribution and we have the obvious.

COROLLARY:

$$c_{0,2n} > c_{1,2n} > c_{2,2n} > \cdots > c_{n-1,2n}. \tag{5.4}$$

6. THE EXPECTATIONS.

THEOREM 4. *Let Z_{2n} and C_{2n} denote, respectively, the number of zeros and the number of changes of sign among the terms $S_1, \ldots,$*

[3] See, for example, formula (9.11) of Chapter 2 of the book quoted above.

S_{2n-1}. *For the expectations of these random variables we have*

$$2\,E(C_{2n}) = E(Z_{2n}) = 2nu_{2n} - 1 \tag{6.1}$$

and thus

$$2\,E(C_{2n}) = E(Z_{2n}) \sim \left(\frac{2}{\pi}\right)^{1|} (2n)^{\frac{1}{2}} \quad as\ n \longrightarrow \infty\ . \tag{6.2}$$

(These formulas shows that the density of the zeros and of changes of sign decreses at a fast rate.)

Proof. Define new random variables by $Y_j = 1$ if $X_j = 0$, and $Y_j = 0$ if $X_j \neq 0$. Then

$$2\,E(C_{2n}) = E(Z_{2n}) = \sum_{j=1}^{2n-1} E(Y_j) = \sum_{r=1}^{n-1} u_{2r}\ . \tag{6.3}$$

and (6.1) follows by induction.

7. Later returns to the origin.

As a further application of the present elementary approach let us prove an important formula half of which has been proved by rather involved analytical methods [4].

Theorem 5. *Let $f_{k,2n}$ denote the probability that the k-th return to the origin takes place at the $2n$-th step (that is, $f_{k,2n}$ is the probability that $S_{2n} = 0$ and exactly $r - 1$ among the S_j with $1 \leq j < 2n$ vanish). Then*

$$f_{k,2n} = z_{k,2n} - z_{k+1,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n} \frac{k}{2n-k}\ . \tag{7.1}$$

Proof. It is clear that $f_{1,2n} = f_{2n}$ and that the $f_{k,2n}$ satisfy the recurrence relation (4.9) and hence also (4.11). If we define $f_{0,2n} = 0$ for n 1 and $f_{0,0} = 1$, then (7.1) is true for $k = 0,1$ and therefore for all $k \geq 0$.

Reçu le 17 mai 1957.

[4] See, for example, *ibid.*, formula (6.15) of Chapter 12.