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of the sequence we can speak of changes of sign only at the places $j \leq 2n - 2$.

THEOREM 3. *Let $c_{r, 2n}$ denote the probability that there exist exactly r indices j such that*

$$(5.1) \quad S_{j-1} S_{j+1} < 0, \quad 1 \leq j \leq 2n - 1.$$

Then

$$(5.2) \quad c_{r, 2n} = \frac{1}{2^{2n-2}} \binom{2n-1}{n-1-r}.$$

Proof. Let us say that two sequences S_1, \dots, S_m and S'_1, \dots, S'_m are *similar* if $|S_j| = |S'_j|$ for $j = 1, 2, \dots, m$. Obviously $-S_1, -S_2, \dots, -S_{2n}$ represents the only sequence similar to S_1, \dots, S_{2n} and such that changes of sign occur at the same places. On the other hand, if exactly k among the terms S_1, \dots, S_{2n-2} vanish, there exist exactly 2^{k+1} sequences similar to the sequence S_1, \dots, S_{2n} . Out of k places we may choose r places in $\binom{k}{r}$ different ways, and it is therefore seen that

$$(5.3) \quad \begin{aligned} c_{r, 2n} &= 2 \sum_{k=r}^{n-1} \binom{k}{r} 2^{-(k+1)} z_{k, 2n-2} \\ &= \frac{1}{2^{2n-2}} \sum_{k=r}^{n-1} \binom{k}{r} \binom{2n-2-k}{n-1} \end{aligned}$$

A well-known formula for binomial coefficients³ which can be proved by induction now shows that (5.2) is true.

In (5.2) we recognize the binomial distribution and we have the obvious.

COROLLARY:

$$(5.4) \quad c_{0, 2n} > c_{1, 2n} > c_{2, 2n} > \dots > c_{n-1, 2n}.$$

6. THE EXPECTATIONS.

THEOREM 4. *Let Z_{2n} and C_{2n} denote, respectively, the number of zeros and the number of changes of sign among the terms $S_1, \dots,$*

³ See, for example, formula (9.14) of Chapter 2 of the book quoted above.

S_{2n-1} . For the expectations of these random variables we have

$$(6.1) \quad 2 E (C_{2n}) = E (Z_{2n}) = 2nu_{2n} - 1$$

and thus

$$(6.2) \quad 2 E (C_{2n}) = E (Z_{2n}) \sim \left(\frac{2}{\pi}\right)^{1/2} (2n)^{1/2} \text{ as } n \rightarrow \infty .$$

(These formulas shows that the density of the zeros and of changes of sign decreases at a fast rate.)

Proof. Define new random variables by $Y_j = 1$ if $X_j = 0$, and $Y_j = 0$ if $X_j \neq 0$. Then

$$(6.3) \quad 2 E (C_{2n}) = E (Z_{2n}) = \sum_{j=1}^{2n-1} E (Y_j) = \sum_{r=1}^{n-1} u_{2r} .$$

and (6.1) follows by induction.

7. LATER RETURNS TO THE ORIGIN.

As a further application of the present elementary approach let us prove an important formula half of which has been proved by rather involved analytical methods ⁴.

THEOREM 5. Let $f_{k,2n}$ denote the probability that the k -th return to the origin takes place at the $2n$ -th step (that is, $f_{k,2n}$ is the probability that $S_{2n} = 0$ and exactly $r - 1$ among the S_j with $1 \leq j < 2n$ vanish). Then

$$(7.1) \quad f_{k,2n} = z_{k,2n} - z_{k+1,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n} \frac{k}{2n-k} .$$

Proof. It is clear that $f_{1,2n} = f_{2n}$ and that the $f_{k,2n}$ satisfy the recurrence relation (4.9) and hence also (4.11). If we define $f_{0,2n} = 0$ for $n \geq 1$ and $f_{0,0} = 1$, then (7.1) is true for $k = 0, 1$ and therefore for all $k \geq 0$.

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⁴ See, for example, *ibid.*, formula (6.15) of Chapter 12.