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of the sequence we can speak of changes of sign only at the places  $j \leq 2n - 2$ .

**THEOREM 3.** *Let  $c_{r,2n}$  denote the probability that there exist exactly  $r$  indices  $j$  such that*

$$(5.1) \quad S_{j-1} S_{j+1} < 0, \quad 1 \leq j \leq 2n - 1.$$

*Then*

$$(5.2) \quad c_{r,2n} = \frac{1}{2^{2n-2}} \binom{2n-1}{n-1-r}.$$

*Proof.* Let us say that two sequences  $S_1, \dots, S_m$  and  $S'_1, \dots, S'_m$  are *similar* if  $|S_j| = |S'_j|$  for  $j = 1, 2, \dots, m$ . Obviously  $-S_1, -S_2, \dots, -S_{2n}$  represents the only sequence similar to  $S_1, \dots, S_{2n}$  and such that changes of sign occur at the same places. On the other hand, if exactly  $k$  among the terms  $S_1, \dots, S_{2n-2}$  vanish, there exist exactly  $2^{k+1}$  sequences similar to the sequence  $S_1, \dots, S_{2n}$ . Out of  $k$  places we may choose  $r$  places in  $\binom{k}{r}$  different ways, and it is therefore seen that

$$(5.3) \quad \begin{aligned} c_{r,2n} &= 2 \sum_{k=r}^{n-1} \binom{k}{r} 2^{-(k+1)} z_{k,2n-2} \\ &= \frac{1}{2^{2n-2}} \sum_{k=r}^{n-1} \binom{k}{r} \binom{2n-2-k}{n-1}. \end{aligned}$$

A well-known formula for binomial coefficients<sup>3</sup> which can be proved by induction now shows that (5.2) is true.

In (5.2) we recognize the binomial distribution and we have the obvious.

**COROLLARY:**

$$(5.4) \quad c_{0,2n} > c_{1,2n} > c_{2,2n} > \dots > c_{n-1,2n}.$$

## 6. THE EXPECTATIONS.

**THEOREM 4.** *Let  $Z_{2n}$  and  $C_{2n}$  denote, respectively, the number of zeros and the number of changes of sign among the terms  $S_1, \dots,$*

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<sup>3</sup> See, for example, formula (9.11) of Chapter 2 of the book quoted above.

$S_{2n-1}$ . For the expectations of these random variables we have

$$(6.1) \quad 2E(C_{2n}) = E(Z_{2n}) = 2nu_{2n} - 1$$

and thus

$$(6.2) \quad 2E(C_{2n}) = E(Z_{2n}) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (2n)^{\frac{1}{2}} \quad \text{as } n \rightarrow \infty.$$

(These formulas shows that the density of the zeros and of changes of sign decreases at a fast rate.)

*Proof.* Define new random variables by  $Y_j = 1$  if  $X_j = 0$ , and  $Y_j = 0$  if  $X_j \neq 0$ . Then

$$(6.3) \quad 2E(C_{2n}) = E(Z_{2n}) = \sum_{j=1}^{2n-1} E(Y_j) = \sum_{r=1}^{n-1} u_{2r}.$$

and (6.1) follows by induction.

## 7. LATER RETURNS TO THE ORIGIN.

As a further application of the present elementary approach let us prove an important formula half of which has been proved by rather involved analytical methods <sup>4</sup>.

**THEOREM 5.** Let  $f_{k,2n}$  denote the probability that the  $k$ -th return to the origin takes place at the  $2n$ -th step (that is,  $f_{k,2n}$  is the probability that  $S_{2n} = 0$  and exactly  $r - 1$  among the  $S_j$  with  $1 \leq j < 2n$  vanish). Then

$$(7.1) \quad f_{k,2n} = z_{k,2n} - z_{k+1,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n} \frac{k}{2n-k}.$$

*Proof.* It is clear that  $f_{1,2n} = f_{2n}$  and that the  $f_{k,2n}$  satisfy the recurrence relation (4.9) and hence also (4.11). If we define  $f_{0,2n} = 0$  for  $n \geq 1$  and  $f_{0,0} = 1$ , then (7.1) is true for  $k = 0, 1$  and therefore for all  $k \geq 0$ .

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<sup>4</sup> See, for example, *ibid.*, formula (6.15) of Chapter 12.