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Comparing (4.5) and (4.7) we see that

$$(4.8) \quad z_{1,2n} = u_{2n} = z_{0,2n} \quad \text{for } n \geq 1.$$

In like manner we can calculate  $z_{2,2n}, z_{3,2n}, \dots$  from the recursion formula

$$(4.9) \quad z_{k,2n} = \sum_{r=1}^{n-1} f_{2r} z_{k-1,2n-2r}, \quad k \geq 2, \quad n \geq 1.$$

which is proved exactly as (4.5). For  $k \geq 2$  the right side differs from the right side in (4.5) only in that the term  $r = n$  is absent, and therefore

$$(4.10) \quad z_{k,2n} = z_{1,2n} - f_{2n} = 2z_{1,2n} - z_{0,2n-2}, \quad n \geq 1.$$

From the last two relations we see directly by induction that *for  $k \geq 2$  and  $n \geq 1$  we have the recursion formula*

$$(4.11) \quad z_{k,2n} = 2z_{k-1,2n} - z_{k-2,2n-2}$$

If we write  $z_{k,2n} = 2^{k-2n} a_{k,2n}$  then (4.11) reduces to

$$(4.12) \quad a_{k-1,2n} = a_{k,2n} + a_{k-2,2n-2}$$

which is the well-known addition relation for binomial coefficients, and thus (4.2) holds.

This theorem has the following surprising

COROLLARY. *For each  $n \geq 1$  we have*

$$(4.13) \quad z_{0,2n} = z_{1,2n} > z_{2,2n} > z_{3,2n} > \dots > z_{n,2n}$$

Thus, independently of the number  $n$  of steps, the *most probable number of zeros is 0*, and the smaller the number, the more probable it is.

## 5. THE NUMBER OF CHANGES OF SIGN.

We say that in the sequence  $S_1, \dots, S_{2n}$  a *change of sign occurs at the place  $j$*  if  $S_{j-1}$  and  $S_{j+1}$  are of opposite signs. This requires that  $S_j = 0$ , and so  $j$  must be even. Given the first  $2n$  terms

of the sequence we can speak of changes of sign only at the places  $j \leq 2n - 2$ .

**THEOREM 3.** *Let  $c_{r,2n}$  denote the probability that there exist exactly  $r$  indices  $j$  such that*

$$(5.1) \quad S_{j-1} S_{j+1} < 0, \quad 1 \leq j \leq 2n - 1.$$

*Then*

$$(5.2) \quad c_{r,2n} = \frac{1}{2^{2n-2}} \binom{2n-1}{n-1-r}.$$

*Proof.* Let us say that two sequences  $S_1, \dots, S_m$  and  $S'_1, \dots, S'_m$  are *similar* if  $|S_j| = |S'_j|$  for  $j = 1, 2, \dots, m$ . Obviously  $-S_1, -S_2, \dots, -S_{2n}$  represents the only sequence similar to  $S_1, \dots, S_{2n}$  and such that changes of sign occur at the same places. On the other hand, if exactly  $k$  among the terms  $S_1, \dots, S_{2n-2}$  vanish, there exist exactly  $2^{k+1}$  sequences similar to the sequence  $S_1, \dots, S_{2n}$ . Out of  $k$  places we may choose  $r$  places in  $\binom{k}{r}$  different ways, and it is therefore seen that

$$(5.3) \quad \begin{aligned} c_{r,2n} &= 2 \sum_{k=r}^{n-1} \binom{k}{r} 2^{-(k+1)} z_{k,2n-2} \\ &= \frac{1}{2^{2n-2}} \sum_{k=r}^{n-1} \binom{k}{r} \binom{2n-2-k}{n-1}. \end{aligned}$$

A well-known formula for binomial coefficients<sup>3</sup> which can be proved by induction now shows that (5.2) is true.

In (5.2) we recognize the binomial distribution and we have the obvious.

**COROLLARY:**

$$(5.4) \quad c_{0,2n} > c_{1,2n} > c_{2,2n} > \dots > c_{n-1,2n}.$$

## 6. THE EXPECTATIONS.

**THEOREM 4.** *Let  $Z_{2n}$  and  $C_{2n}$  denote, respectively, the number of zeros and the number of changes of sign among the terms  $S_1, \dots,$*

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<sup>3</sup> See, for example, formula (9.11) of Chapter 2 of the book quoted above.