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Comparing (4.5) and (4.7) we see that

$$(4.8) \quad z_{1,2n} = u_{2n} = z_{0,2n} \quad \text{for } n \geq 1.$$

In like manner we can calculate $z_{2,2n}, z_{3,2n}, \dots$ from the recursion formula

$$(4.9) \quad z_{k,2n} = \sum_{r=1}^{n-1} f_{2r} z_{k-1,2n-2r}, \quad k \geq 2, \quad n \geq 1.$$

which is proved exactly as (4.5). For $k \geq 2$ the right side differs from the right side in (4.5) only in that the term $r = n$ is absent, and therefore

$$(4.10) \quad z_{k,2n} = z_{1,2n} - f_{2n} = 2z_{1,2n} - z_{0,2n-2}, \quad n \geq 1.$$

From the last two relations we see directly by induction that for $k \geq 2$ and $n \geq 1$ we have the recursion formula

$$(4.11) \quad z_{k,2n} = 2z_{k-1,2n} - z_{k-2,2n-2}$$

If we write $z_{k,2n} = 2^{k-2n} a_{k,2n}$ then (4.11) reduces to

$$(4.12) \quad a_{k-1,2n} = a_{k,2n} + a_{k-2,2n-2}$$

which is the well-known addition relation for binomial coefficients, and thus (4.2) holds.

This theorem has the following surprising

COROLLARY. For each $n \geq 1$ we have

$$(4.13) \quad z_{0,2n} = z_{1,2n} > z_{2,2n} > z_{3,2n} > \dots > z_{n,2n}$$

Thus, independently of the number n of steps, the *most probable number of zeros* is 0, and the smaller the number, the more probable it is.

5. THE NUMBER OF CHANGES OF SIGN.

We say that in the sequence S_1, \dots, S_{2n} a *change of sign occurs at the place j* if S_{j-1} and S_{j+1} are of opposite signs. This requires that $S_j = 0$, and so j must be even. Given the first $2n$ terms

of the sequence we can speak of changes of sign only at the places $j \leq 2n - 2$.

THEOREM 3. *Let $c_{r, 2n}$ denote the probability that there exist exactly r indices j such that*

$$(5.1) \quad S_{j-1} S_{j+1} < 0, \quad 1 \leq j \leq 2n - 1.$$

Then

$$(5.2) \quad c_{r, 2n} = \frac{1}{2^{2n-2}} \binom{2n-1}{n-1-r}.$$

Proof. Let us say that two sequences S_1, \dots, S_m and S'_1, \dots, S'_m are *similar* if $|S_j| = |S'_j|$ for $j = 1, 2, \dots, m$. Obviously $-S_1, -S_2, \dots, -S_{2n}$ represents the only sequence similar to S_1, \dots, S_{2n} and such that changes of sign occur at the same places. On the other hand, if exactly k among the terms S_1, \dots, S_{2n-2} vanish, there exist exactly 2^{k+1} sequences similar to the sequence S_1, \dots, S_{2n} . Out of k places we may choose r places in $\binom{k}{r}$ different ways, and it is therefore seen that

$$(5.3) \quad \begin{aligned} c_{r, 2n} &= 2 \sum_{k=r}^{n-1} \binom{k}{r} 2^{-(k+1)} z_{k, 2n-2} \\ &= \frac{1}{2^{2n-2}} \sum_{k=r}^{n-1} \binom{k}{r} \binom{2n-2-k}{n-1} \end{aligned}$$

A well-known formula for binomial coefficients³ which can be proved by induction now shows that (5.2) is true.

In (5.2) we recognize the binomial distribution and we have the obvious.

COROLLARY:

$$(5.4) \quad c_{0, 2n} > c_{1, 2n} > c_{2, 2n} > \dots > c_{n-1, 2n}.$$

6. THE EXPECTATIONS.

THEOREM 4. *Let Z_{2n} and C_{2n} denote, respectively, the number of zeros and the number of changes of sign among the terms $S_1, \dots,$*

³ See, for example, formula (9.14) of Chapter 2 of the book quoted above.