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Artikel: THE NUMBERS OF ZEROS AND OF CHANGES OF SIGN IN A SYMMETRIC RANDOM WALK

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Comparing (4.5) and (4.7) we see that

(4.8) $$z_{1,2n} = u_{2n} = z_{0,2n} \qquad \textit{for } n \geq 1\ .$$

In like manner we can calculate $z_{2,\,2n}, z_{3,\,2n}, \ldots$ from the recursion formula

(4.9) $$z_{k,2n} = \sum_{r=1}^{n-1} f_{2r}\, z_{k-1,\,2n-2r}\ , \qquad k \geq 2, \quad n \geq 1\ .$$

which is proved exactly as (4.5). For $k \geq 2$ the right side differs from the right side in (4.5) only in that the term $r = n$ is absent, and therefore

(4.10) $$z_{k,\,2n} = z_{1,2n} - f_{2n} = 2z_{1,2n} - z_{0,\,2n-2}\ , \qquad n \geq 1\ .$$

From the last two relations we see directly by induction that *for* $k \geq 2$ *and* $n \geq 1$ *we have the recursion formula*

(4.11) $$z_{k,\,2n} = 2z_{k-1,\,2n} - z_{k-2,\,2n-2}$$

If we write $z_{k,\,2n} = 2^{k-2n}\, a_{k,\,2n}$ then (4.11) reduces to

(4.12) $$a_{k-1,\,2n} = a_{k,\,2n} + a_{k-2,\,2n-2}$$

which is the well-known addition relation for binomial coefficients, and thus (4.2) holds.

This theorem has the following surprising

COROLLARY. *For each* $n \geq 1$ *we have*

(4.13) $$z_{0,\,2n} = z_{1,\,2n} > z_{2,\,2n} > z_{3,\,2n} > \cdots > z_{n,\,2n}$$

Thus, independently of the number n of steps, the *most probable number of zeros is* 0, and the smaller the number, the more probable it is.

5. THE NUMBER OF CHANGES OF SIGN.

We say that in the sequence $S_1, \ldots, S_{2n}$ *a change of sign occurs at the place* j if S_{j-1} and S_{j+1} are of opposite signs. This requires that $S_j = 0$, and so j must be even. Given the first $2n$ terms of the sequence we can speak of changes of sign only at the places $j \leq 2n-2$.

THEOREM 3. *Let $c_{r,2n}$ denote the probability that there exist exactly r indices j such that*

$$S_{j-1}\, S_{j+1} < 0\,, \qquad 1 \leq j \leq 2n-1\,. \tag{5.1}$$

Then

$$c_{r,2n} = \frac{1}{2^{2n-2}} \binom{2n-1}{n-1-r}\,. \tag{5.2}$$

Proof. Let us say that two sequences $S_1, \ldots, S_m$ and $S'_1, \ldots, S'_m$ are *similar* if $|S_j| = |S'_j|$ for $j = 1, 2, \ldots, m$. Obviously $-S_1, -S_2, \ldots, -S_{2n}$ represents the only sequence similar to $S_1, \ldots, S_{2n}$ and such that changes of sign occur at the same places. On the other hand, if exactly k among the terms $S_1, \ldots, S_{2n-2}$ vanish, there exist exactly 2^{k+1} sequences similar to the sequence $S_1, \ldots, S_{2n}$. Out of k places we may choose r places in $\binom{k}{r}$ different ways, and it is therefore seen that

$$\begin{aligned} c_{r,2n} &= 2 \sum_{k=r}^{n-1} \binom{k}{r} 2^{-(k+1)} z_{k,2n-2} \\ &= \frac{1}{2^{2n-2}} \sum_{k=r}^{n-1} \binom{k}{r} \binom{2n-2-k}{n-1}\,. \end{aligned} \tag{5.3}$$

A well-known formula for binomial coefficients [3] which can be proved by induction now shows that (5.2) is true.

In (5.2) we recognize the binomial distribution and we have the obvious.

COROLLARY:

$$c_{0,2n} > c_{1,2n} > c_{2,2n} > \cdots > c_{n-1,2n}\,. \tag{5.4}$$

6. THE EXPECTATIONS.

THEOREM 4. *Let Z_{2n} and C_{2n} denote, respectively, the number of zeros and the number of changes of sign among the terms $S_1, \ldots,$*

[3] See, for example, formula (9.11) of Chapter 2 of the book quoted above.