

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 3 (1957)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE NUMBERS OF ZEROS AND OF CHANGES OF SIGN IN A SYMMETRIC RANDOM WALK
Autor: Feller, William
Kapitel: 4. The number of zeros.
DOI: <https://doi.org/10.5169/seals-33746>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 29.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

4. THE NUMBER OF ZEROS.

THEOREM 2. For $n \geq 1$ let $z_{k,n}$ be the probability that exactly k among the n partial sums S_1, \dots, S_n vanish. For $n = 0$ put

$$(4.1) \quad z_{0,0} = 1, \quad z_{1,0} = z_{2,0} = \dots = 0.$$

Then

$$(4.2) \quad z_{k,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n}.$$

Proof. By definition

$$(4.3) \quad z_{0,2n} = p_{2n} = u_{2n}, \quad (n \geq 0).$$

To evaluate $z_{1,2n}$ denote by B_r the event that among the partial sums S_1, \dots, S_{2n} exactly one vanishes and its index equals $2r$. Then for $r < n$

$$\begin{aligned} B_r &= \{ S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2r+1} \neq 0, \dots, S_{2n} \neq 0 \} \\ &\equiv \{ S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0 \} \cap \{ S_{2r+1} - S_{2r} \neq 0, \dots, S_{2n} - S_{2r} \neq 0 \}. \end{aligned}$$

Since the two events on the right are stochastically independent and the B_r are mutually exclusive we conclude that

$$(4.5) \quad z_{1,2n} = \sum_{r=1}^n P \{ B_r \} = \sum_{r=1}^n f_{2r} z_{0,2n-2r}.$$

Now by Theorem 1 the last event on the right in (4.4) has the same probability as the event $\{ S_{2n} - S_{2r} = 0 \}$ and hence we have for $r \leq n$

$$(4.6) \quad P \{ B_r \} = P \{ S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2n} = 0 \}.$$

The events appearing on the right side are mutually exclusive and their union is the event $\{ S_{2n} = 0 \}$; hence

$$(4.7) \quad \sum_{r=1}^n P \{ B_r \} = P \{ S_{2n} = 0 \} = u_{2n}.$$

Comparing (4.5) and (4.7) we see that

$$(4.8) \quad z_{1,2n} = u_{2n} = z_{0,2n} \quad \text{for } n \geq 1.$$

In like manner we can calculate $z_{2,2n}, z_{3,2n}, \dots$ from the recursion formula

$$(4.9) \quad z_{k,2n} = \sum_{r=1}^{n-1} f_{2r} z_{k-1,2n-2r}, \quad k \geq 2, \quad n \geq 1.$$

which is proved exactly as (4.5). For $k \geq 2$ the right side differs from the right side in (4.5) only in that the term $r = n$ is absent, and therefore

$$(4.10) \quad z_{k,2n} = z_{1,2n} - f_{2n} = 2z_{1,2n} - z_{0,2n-2}, \quad n \geq 1.$$

From the last two relations we see directly by induction that for $k \geq 2$ and $n \geq 1$ we have the recursion formula

$$(4.11) \quad z_{k,2n} = 2z_{k-1,2n} - z_{k-2,2n-2}$$

If we write $z_{k,2n} = 2^{k-2n} a_{k,2n}$ then (4.11) reduces to

$$(4.12) \quad a_{k-1,2n} = a_{k,2n} + a_{k-2,2n-2}$$

which is the well-known addition relation for binomial coefficients, and thus (4.2) holds.

This theorem has the following surprising

COROLLARY. For each $n \geq 1$ we have

$$(4.13) \quad z_{0,2n} = z_{1,2n} > z_{2,2n} > z_{3,2n} > \dots > z_{n,2n}$$

Thus, independently of the number n of steps, the *most probable number of zeros* is 0, and the smaller the number, the more probable it is.

5. THE NUMBER OF CHANGES OF SIGN.

We say that in the sequence S_1, \dots, S_{2n} a *change of sign occurs at the place j* if S_{j-1} and S_{j+1} are of opposite signs. This requires that $S_j = 0$, and so j must be even. Given the first $2n$ terms