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## 4. The number of zeros.

THEOREM 2. *For $n \geq 1$ let $z_{k,n}$ be the probability that exactly $k$ among the $n$ partial sums $S_1, \ldots, S_n$ vanish. For $n = 0$ put*

$$z_{0,0} = 1, \qquad z_{1,0} = z_{2,0} = \ldots = 0. \tag{4.1}$$

*Then*

$$z_{k,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n}. \tag{4.2}$$

*Proof.* By definition

$$z_{0,2n} = p_{2n} = u_{2n}, \qquad (n \geq 0). \tag{4.3}$$

To evaluate $z_{1,2n}$ denote by $B_r$ the event that among the partial sums $S_1, \ldots, S_{2n}$ exactly one vanishes and its index equals $2r$. Then for $r < n$

$$B_r = \{ S_1 \neq 0, \ldots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2r+1} \neq 0, \ldots, S_{2n} \neq 0 \}$$
$$\equiv \{ S_1 \neq 0, \ldots, S_{2r-1} \neq 0, S_{2r} = 0 \} \cap \{ S_{2r+1} - S_{2r} \neq 0, \ldots, S_{2n} - S_{2r} \neq 0 \}.$$

Since the two events on the right are stochastically independent and the $B_r$ are mutually exclusive we conclude that

$$z_{1,2n} = \sum_{r=1}^{n} P\{ B_r \} = \sum_{r=1}^{n} f_{2r} z_{0,2n-2r}. \tag{4.5}$$

Now by Theorem 1 the last event on the right in (4.4) has the same probability as the event $\{ S_{2n} - S_{2r} = 0 \}$ and hence we have for $r \leq n$

$$P\{ B_r \} = P\{ S_1 \neq 0, \ldots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2n} = 0 \}. \tag{4.6}$$

The events appearing on the right side are mutually exclusive and their union is the event $\{ S_{2n} = 0 \}$; hence

$$\sum_{r=1}^{n} P\{ B_r \} = P\{ S_{2n} = 0 \} = u_{2n}. \tag{4.7}$$

Comparing (4.5) and (4.7) we see that

$$z_{1,2n} = u_{2n} = z_{0,2n} \qquad \textit{for } n \geq 1 . \tag{4.8}$$

In like manner we can calculate $z_{2,\,2n}, z_{3,\,2n}, \ldots$ from the recursion formula

$$z_{k,2n} = \sum_{r=1}^{n-1} f_{2r}\, z_{k-1,2n-2r} , \qquad k \geq 2, \quad n \geq 1 . \tag{4.9}$$

which is proved exactly as (4.5). For $k \geq 2$ the right side differs from the right side in (4.5) only in that the term $r = n$ is absent, and therefore

$$z_{k,2n} = z_{1,2n} - f_{2n} = 2 z_{1,2n} - z_{0,2n-2} , \qquad n \geq 1 . \tag{4.10}$$

From the last two relations we see directly by induction that *for $k \geq 2$ and $n \geq 1$ we have the recursion formula*

$$z_{k,2n} = 2 z_{k-1,2n} - z_{k-2,2n-2} \tag{4.11}$$

If we write $z_{k,2n} = 2^{k-2n} a_{k,2n}$ then (4.11) reduces to

$$a_{k-1,2n} = a_{k,2n} + a_{k-2,2n-2} \tag{4.12}$$

which is the well-known addition relation for binomial coefficients, and thus (4.2) holds.

This theorem has the following surprising

COROLLARY. *For each $n \geq 1$ we have*

$$z_{0,2n} = z_{1,2n} > z_{2,2n} > z_{3,2n} > \cdots > z_{n,2n} \tag{4.13}$$

Thus, independently of the number $n$ of steps, the *most probable number of zeros is* 0, and the smaller the number, the more probable it is.

## 5. THE NUMBER OF CHANGES OF SIGN.

We say that in the sequence $S_1, \ldots, S_{2n}$ *a change of sign occurs at the place* $j$ if $S_{j-1}$ and $S_{j+1}$ are of opposite signs. This requires that $S_j = 0$, and so $j$ must be even. Given the first $2n$ terms