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3. The probability of no zeros.

Let p_n *be the probability that the first n steps do not lead to a return to the origin, that is,*

$$p_n = \mathrm{P}\{ S_1 \neq 0 , \quad S_2 \neq 0, \ldots, S_n \neq 0 \}, \qquad p_0 = 1 . \tag{3.1}$$

Theorem 1. *We have*

$$p_{2n} = u_{2n} \qquad n = 0, 1, 2, \ldots \tag{3.2}$$

Proof. Denote by A_r the event that among the partial sums $S_0, S_1, \ldots, S_{2n}$ the *last zero* has index $2r$:

$$A_r = \{ S_{2r} = 0, \quad S_{2r+1} \neq 0, \quad S_{2r+2} \neq 0, \ldots, \quad S_{2n} \neq 0 \} \tag{3.3}$$

$$= \{ S_{2r} = 0 \} \cap \{ S_{2r+1} - S_{2r} \neq 0 , \quad S_{2r+2} - S_{2r} \neq 0, \ldots, \quad S_{2n} - S_{2r} \neq 0 \}$$

The two events on the extreme right side are independent and have, respectively, probabilities u_{2r} and p_{2n-2r}. The union of the events A_r covers the sample space of the sequences $S_0, \ldots, S_n$, and these events are mutually exclusive. Therefore

$$1 = \sum_{r=0}^{n} \mathrm{P}\{ A_r \} = \sum_{r=0}^{n} u_{2r}\, p_{2n-2r} \tag{3.4}$$

and a comparison of (3.4) with (3.1) proves the theorem.

The last theorem is fully equivalent to the following corollary which is well known.

Corollary. *Let* f_{2n} *be the probability that the first return to the origin takes place at the 2n-th step, that is,*

$$f_{2n} = \mathrm{P}\{ S_1 \neq 0, \quad S_2 \neq 0, \ldots, S_{2n-1} \neq 0, S_{2n} = 0 \} . \tag{3.5}$$

Then

$$f_{2n} = u_{2n} - u_{2n-2} . \tag{3.6}$$

Proof. From (3.1) and (3.5) it is obvious that $f_{2n} = u_{2n} - u_{2n-2}$.