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Furthermore, the derivation is of a quite elementary nature and therefore of some independent interest. In fact, we shall start from the simple combinatorial formula (2.4) and from it derive all results by a direct procedure without presupposing any knowledge concerning random walks and without using any analytical tools.

2. Preparations.

Let X_1, X_2, ... denote an infinite sequence of mutually independent random variables each assuming the values ± 1 with probability $\frac{1}{2}$. Put

$$(2.1) \qquad S_0 = 1, \qquad S_n = X_1 + X_2 + \dots + X_n \qquad (n \geq 1)$$

Then S_n is to be interpreted as the coordinate, at time n (or after n steps), in a one-dimensional symmetric random walk starting from the origin. *A return to the origin occurs at time n if* $S_n = 0$. Obviously n must be even. For the probability of such a return we write

$$(2.2) \qquad u_n = P\{S_n = 0\}, \qquad u_0 = 1$$

Clearly

$$(2.3) \qquad u_{2n} = \frac{1}{2^{2n}}\binom{2n}{n}, \qquad u_{2n-1} = 0 .$$

All our considerations will depend on the following simple and well known Lemma:

$$(2.4) \qquad \sum_{r=0}^{n} u_{2r}\, u_{2n-2r} = 1 .$$

Proof. We introduce the generating function

$$(2.5) \qquad U(s) = \sum_{n=0}^{\infty} u_{2n}\, s^{2n} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} s^{2n} = (1 - s^2)^{-\frac{1}{2}} .$$

It is then clear that the left side on (2.4) equals the coefficient of s^{2n} in $U^2(s) = (1 - s^2)^{-1}$, and thus (2.4) is true.