

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	3 (1957)
<b>Heft:</b>	1: L'ENSEIGNEMENT MATHÉMATIQUE
 <b>Artikel:</b>	 THE NUMBERS OF ZEROS AND OF CHANGES OF SIGN IN A SYMMETRIC RANDOM WALK
<b>Autor:</b>	Feller, William
<b>Kapitel:</b>	1. Introduction.
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-33746">https://doi.org/10.5169/seals-33746</a>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# THE NUMBERS OF ZEROS AND OF CHANGES OF SIGN IN A SYMMETRIC RANDOM WALK \*

BY

William FELLER, Princeton University

## 1. INTRODUCTION.

After a very long period of oblivion, the theory of the symmetric random walk once more attracts widespread attention. Curious and totally unexpected fluctuation phenomena have been discovered and described in the arc sine law and other equally preposterous theorems<sup>1</sup>. As E. Sparre Andersen has shown<sup>2</sup>, these laws apply to an exceedingly large class of chance processes and they have completely revolutionized our notions concerning chance fluctuations when cumulative effects are involved.

Most of these newly discovered theorems are related to the consecutive returns to the origin. The present paper has the modest purpose of deriving *explicit formulas for the probability distributions of the number of returns to the origin, the number of changes of sign, etc.* during the first  $n$  steps of a symmetric random walk in one dimension. The limiting form of these distributions as  $n \rightarrow \infty$  are known, and therefore no special importance can be ascribed to the knowledge of the explicit formulas. However, they are pleasing and surprisingly simple.

---

\* This research was done while the author was on sabbatical leave of absence with the support of the United States Air Force through the Air Force Office of Scientific Research and Development Command under Contract No. AF18 (603)-24 with Princeton University.

<sup>1</sup> See, for example, *An Introduction to Probability Theory and its Applications* by William FELLER, New York, 1950, Chapter 12. A more exhaustive treatment will be contained in Chapter III of the second edition of this book.

<sup>2</sup> On the fluctuations of sums of random variables I and II, *Mathematica Scandinavica*, vol. 1 (1953), pp. 263-285 and vol. 2 (1954), pp. 195-223.

Furthermore, the derivation is of a quite elementary nature and therefore of some independent interest. In fact, we shall start from the simple combinatorial formula (2.4) and from it derive all results by a direct procedure without presupposing any knowledge concerning random walks and without using any analytical tools.

## 2. PREPARATIONS.

Let  $X_1, X_2, \dots$  denote an infinite sequence of mutually independent random variables each assuming the values  $\pm 1$  with probability  $\frac{1}{2}$ . Put

$$(2.1) \quad S_0 = 1, \quad S_n = X_1 + X_2 + \dots + X_n \quad (n \geq 1)$$

Then  $S_n$  is to be interpreted as the coordinate, at time  $n$  (or after  $n$  steps), in a one-dimensional symmetric random walk starting from the origin. *A return to the origin occurs at time  $n$  if  $S_n = 0$ .* Obviously  $n$  must be even. For the probability of such a return we write

$$(2.2) \quad u_n = P \{ S_n = 0 \}, \quad u_0 = 1$$

Clearly

$$(2.3) \quad u_{2n} = \frac{1}{2^{2n}} \binom{2n}{n}, \quad u_{2n-1} = 0.$$

All our considerations will depend on the following simple and well known LEMMA:

$$(2.4) \quad \sum_{r=0}^n u_{2r} u_{2n-2r} = 1.$$

*Proof.* We introduce the generating function

$$(2.5) \quad U(s) = \sum_{n=0}^{\infty} u_{2n} s^{2n} = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)_n s^{2n} = (1 - s^2)^{-\frac{1}{2}}.$$

It is then clear that the left side on (2.4) equals the coefficient of  $s^{2n}$  in  $U^2(s) = (1 - s^2)^{-1}$ , and thus (2.4) is true.