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THE NUMBERS OF ZEROS AND OF CHANGES OF SIGN IN A SYMMETRIC RANDOM WALK *

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1. INTRODUCTION.

After a very long period of oblivion, the theory of the symmetric random walk once more attracts widespread attention. Curious and totally unexpected fluctuation phenomena have been discovered and described in the arc sine law and other equally preposterous theorems [1]. As E. Sparre Andersen has shown [2], these laws apply to an exceedingly large class of chance processes and they have completely revolutionized our notions concerning chance fluctuations when cumulative effects are involved.

Most of these newly discovered theorems are related to the consecutive returns to the origin. The present paper has the modest purpose of deriving *explicit formulas for the probability distributions of the number of returns to the origin*, *the number of changes of sign*, etc. during the first n steps of a symmetric random walk in one dimension. The limiting form of these distributions as $n \longrightarrow \infty$ are known, and therefore no special importance can be ascribed to the knowledge of the explicit formulas. However, they are pleasing and surprisingly simple.

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[1] See, for example, *An Introduction to Probability Theory and its Applications* by William FELLER, New York, 1950, Chapter 12. A more exhaustive treatment will be contained in Chapter III of the second edition of this book.

[2] On the fluctuations of sums of random variables I and II, *Mathematica Scandinavica*, vol. 1 (1953), pp. 263-285 and vol. 2 (1954), pp. 195-223.

Furthermore, the derivation is of a quite elementary nature and therefore of some independent interest. In fact, we shall start from the simple combinatorial formula (2.4) and from it derive all results by a direct procedure without presupposing any knowledge concerning random walks and without using any analytical tools.

2. Preparations.

Let X_1, X_2, ... denote an infinite sequence of mutually independent random variables each assuming the values ± 1 with probability $\frac{1}{2}$. Put

$$(2.1) \qquad S_0 = 1, \qquad S_n = X_1 + X_2 + \ldots + X_n \qquad (n \geq 1)$$

Then S_n is to be interpreted as the coordinate, at time n (or after n steps), in a one-dimensional symmetric random walk starting from the origin. *A return to the origin occurs at time n if* $S_n = 0$. Obviously n must be even. For the probability of such a return we write

$$(2.2) \qquad u_n = P\{S_n = 0\}, \qquad u_0 = 1$$

Clearly

$$(2.3) \qquad u_{2n} = \frac{1}{2^{2n}}\binom{2n}{n}, \qquad u_{2n-1} = 0 .$$

All our considerations will depend on the following simple and well known Lemma:

$$(2.4) \qquad \sum_{r=0}^{n} u_{2r}\, u_{2n-2r} = 1 .$$

Proof. We introduce the generating function

$$(2.5) \qquad U(s) = \sum_{n=0}^{\infty} u_{2n}\, s^{2n} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} s^{2n} = (1 - s^2)^{-\frac{1}{2}} .$$

It is then clear that the left side on (2.4) equals the coefficient of s^{2n} in $U^2(s) = (1 - s^2)^{-1}$, and thus (2.4) is true.