

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	3 (1957)
<b>Heft:</b>	1: L'ENSEIGNEMENT MATHÉMATIQUE
 <b>Artikel:</b>	 THE NUMBERS OF ZEROS AND OF CHANGES OF SIGN IN A SYMMETRIC RANDOM WALK
<b>Autor:</b>	Feller, William
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-33746">https://doi.org/10.5169/seals-33746</a>

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 16.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# THE NUMBERS OF ZEROS AND OF CHANGES OF SIGN IN A SYMMETRIC RANDOM WALK \*

BY

William FELLER, Princeton University

## 1. INTRODUCTION.

After a very long period of oblivion, the theory of the symmetric random walk once more attracts widespread attention. Curious and totally unexpected fluctuation phenomena have been discovered and described in the arc sine law and other equally preposterous theorems<sup>1</sup>. As E. Sparre Andersen has shown<sup>2</sup>, these laws apply to an exceedingly large class of chance processes and they have completely revolutionized our notions concerning chance fluctuations when cumulative effects are involved.

Most of these newly discovered theorems are related to the consecutive returns to the origin. The present paper has the modest purpose of deriving *explicit formulas for the probability distributions of the number of returns to the origin, the number of changes of sign*, etc. during the first  $n$  steps of a symmetric random walk in one dimension. The limiting form of these distributions as  $n \rightarrow \infty$  are known, and therefore no special importance can be ascribed to the knowledge of the explicit formulas. However, they are pleasing and surprisingly simple.

---

\* This research was done while the author was on sabbatical leave of absence with the support of the United States Air Force through the Air Force Office of Scientific Research and Development Command under Contract No. AF18 (603)-24 with Princeton University.

<sup>1</sup> See, for example, *An Introduction to Probability Theory and its Applications* by William FELLER, New York, 1950, Chapter 12. A more exhaustive treatment will be contained in Chapter III of the second edition of this book.

<sup>2</sup> On the fluctuations of sums of random variables I and II, *Mathematica Scandinavica*, vol. 1 (1953), pp. 263-285 and vol. 2 (1954), pp. 195-223.

Furthermore, the derivation is of a quite elementary nature and therefore of some independent interest. In fact, we shall start from the simple combinatorial formula (2.4) and from it derive all results by a direct procedure without presupposing any knowledge concerning random walks and without using any analytical tools.

## 2. PREPARATIONS.

Let  $X_1, X_2, \dots$  denote an infinite sequence of mutually independent random variables each assuming the values  $\pm 1$  with probability  $\frac{1}{2}$ . Put

$$(2.1) \quad S_0 = 1, \quad S_n = X_1 + X_2 + \dots + X_n \quad (n \geq 1)$$

Then  $S_n$  is to be interpreted as the coordinate, at time  $n$  (or after  $n$  steps), in a one-dimensional symmetric random walk starting from the origin. *A return to the origin occurs at time  $n$  if  $S_n = 0$ .* Obviously  $n$  must be even. For the probability of such a return we write

$$(2.2) \quad u_n = P \{ S_n = 0 \}, \quad u_0 = 1$$

Clearly

$$(2.3) \quad u_{2n} = \frac{1}{2^{2n}} \binom{2n}{n}, \quad u_{2n-1} = 0.$$

All our considerations will depend on the following simple and well known LEMMA:

$$(2.4) \quad \sum_{r=0}^n u_{2r} u_{2n-2r} = 1.$$

*Proof.* We introduce the generating function

$$(2.5) \quad U(s) = \sum_{n=0}^{\infty} u_{2n} s^{2n} = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)_n s^{2n} = (1 - s^2)^{-\frac{1}{2}}.$$

It is then clear that the left side on (2.4) equals the coefficient of  $s^{2n}$  in  $U^2(s) = (1 - s^2)^{-1}$ , and thus (2.4) is true.

## 3. THE PROBABILITY OF NO ZEROS.

Let  $p_n$  be the probability that the first  $n$  steps do not lead to a return to the origin, that is,

$$(3.1) \quad p_n = P \{ S_1 \neq 0, S_2 \neq 0, \dots, S_n \neq 0 \}, \quad p_0 = 1.$$

THEOREM 1. *We have*

$$(3.2) \quad p_{2n} = u_{2n} \quad n = 0, 1, 2, \dots$$

*Proof.* Denote by  $A_r$  the event that among the partial sums  $S_0, S_1, \dots, S_{2n}$  the *last zero* has index  $2r$ :

$$(3.3) \quad A_r = \{ S_{2r} = 0, S_{2r+1} \neq 0, S_{2r+2} \neq 0, \dots, S_{2n} \neq 0 \} \\ = \{ S_{2r} = 0 \} \cap \{ S_{2r+1} - S_{2r} \neq 0, S_{2r+2} - S_{2r} \neq 0, \dots, S_{2n} - S_{2r} \neq 0 \}$$

The two events on the extreme right side are independent and have, respectively, probabilities  $u_{2r}$  and  $p_{2n-2r}$ . The union of the events  $A_r$  covers the sample space of the sequences  $S_0, \dots, S_n$ , and these events are mutually exclusive. Therefore

$$(3.4) \quad 1 = \sum_{r=0}^n P \{ A_r \} = \sum_{r=0}^n u_{2r} p_{2n-2r}$$

and a comparison of (3.4) with (3.1) proves the theorem.

The last theorem is fully equivalent to the following corollary which is well known.

COROLLARY. *Let  $f_{2n}$  be the probability that the first return to the origin takes place at the  $2n$ -th step, that is,*

$$(3.5) \quad f_{2n} = P \{ S_1 \neq 0, S_2 \neq 0, \dots, S_{2n-1} \neq 0, S_{2n} = 0 \}.$$

*Then*

$$(3.6) \quad f_{2n} = u_{2n} - u_{2n-2}.$$

*Proof.* From (3.1) and (3.5) it is obvious that  $f_{2n} = u_{2n} - u_{2n-2}$ .

## 4. THE NUMBER OF ZEROS.

THEOREM 2. For  $n \geq 1$  let  $z_{k,n}$  be the probability that exactly  $k$  among the  $n$  partial sums  $S_1, \dots, S_n$  vanish. For  $n = 0$  put

$$(4.1) \quad z_{0,0} = 1, \quad z_{1,0} = z_{2,0} = \dots = 0.$$

Then

$$(4.2) \quad z_{k,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n}.$$

*Proof.* By definition

$$(4.3) \quad z_{0,2n} = p_{2n} = u_{2n}, \quad (n \geq 0).$$

To evaluate  $z_{1,2n}$  denote by  $B_r$  the event that among the partial sums  $S_1, \dots, S_{2n}$  exactly one vanishes and its index equals  $2r$ . Then for  $r < n$

$$\begin{aligned} B_r &= \{ S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2r+1} \neq 0, \dots, S_{2n} \neq 0 \} \\ &= \{ S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0 \} \cap \{ S_{2r+1} - S_{2r} \neq 0, \dots, S_{2n} - S_{2r} \neq 0 \}. \end{aligned}$$

Since the two events on the right are stochastically independent and the  $B_r$  are mutually exclusive we conclude that

$$(4.5) \quad z_{1,2n} = \sum_{r=1}^n P\{B_r\} = \sum_{r=1}^n f_{2r} z_{0,2n-2r}.$$

Now by Theorem 1 the last event on the right in (4.4) has the same probability as the event  $\{S_{2n} - S_{2r} = 0\}$  and hence we have for  $r \leq n$

$$(4.6) \quad P\{B_r\} = P\{S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2n} = 0\}.$$

The events appearing on the right side are mutually exclusive and their union is the event  $\{S_{2n} = 0\}$ ; hence

$$(4.7) \quad \sum_{r=1}^n P\{B_r\} = P\{S_{2n} = 0\} = u_{2n}.$$

Comparing (4.5) and (4.7) we see that

$$(4.8) \quad z_{1,2n} = u_{2n} = z_{0,2n} \quad \text{for } n \geq 1.$$

In like manner we can calculate  $z_{2,2n}, z_{3,2n}, \dots$  from the recursion formula

$$(4.9) \quad z_{k,2n} = \sum_{r=1}^{n-1} f_{2r} z_{k-1,2n-2r}, \quad k \geq 2, \quad n \geq 1.$$

which is proved exactly as (4.5). For  $k \geq 2$  the right side differs from the right side in (4.5) only in that the term  $r = n$  is absent, and therefore

$$(4.10) \quad z_{k,2n} = z_{1,2n} - f_{2n} = 2z_{1,2n} - z_{0,2n-2}, \quad n \geq 1.$$

From the last two relations we see directly by induction that *for  $k \geq 2$  and  $n \geq 1$  we have the recursion formula*

$$(4.11) \quad z_{k,2n} = 2z_{k-1,2n} - z_{k-2,2n-2}$$

If we write  $z_{k,2n} = 2^{k-2n} a_{k,2n}$  then (4.11) reduces to

$$(4.12) \quad a_{k-1,2n} = a_{k,2n} + a_{k-2,2n-2}$$

which is the well-known addition relation for binomial coefficients, and thus (4.2) holds.

This theorem has the following surprising

COROLLARY. *For each  $n \geq 1$  we have*

$$(4.13) \quad z_{0,2n} = z_{1,2n} > z_{2,2n} > z_{3,2n} > \dots > z_{n,2n}$$

Thus, independently of the number  $n$  of steps, the *most probable number of zeros is 0*, and the smaller the number, the more probable it is.

## 5. THE NUMBER OF CHANGES OF SIGN.

We say that in the sequence  $S_1, \dots, S_{2n}$  a *change of sign occurs at the place  $j$*  if  $S_{j-1}$  and  $S_{j+1}$  are of opposite signs. This requires that  $S_j = 0$ , and so  $j$  must be even. Given the first  $2n$  terms

of the sequence we can speak of changes of sign only at the places  $j \leq 2n - 2$ .

**THEOREM 3.** *Let  $c_{r,2n}$  denote the probability that there exist exactly  $r$  indices  $j$  such that*

$$(5.1) \quad S_{j-1} S_{j+1} < 0, \quad 1 \leq j \leq 2n - 1.$$

*Then*

$$(5.2) \quad c_{r,2n} = \frac{1}{2^{2n-2}} \binom{2n-1}{n-1-r}.$$

*Proof.* Let us say that two sequences  $S_1, \dots, S_m$  and  $S'_1, \dots, S'_m$  are *similar* if  $|S_j| = |S'_j|$  for  $j = 1, 2, \dots, m$ . Obviously  $-S_1, -S_2, \dots, -S_{2n}$  represents the only sequence similar to  $S_1, \dots, S_{2n}$  and such that changes of sign occur at the same places. On the other hand, if exactly  $k$  among the terms  $S_1, \dots, S_{2n-2}$  vanish, there exist exactly  $2^{k+1}$  sequences similar to the sequence  $S_1, \dots, S_{2n}$ . Out of  $k$  places we may choose  $r$  places in  $\binom{k}{r}$  different ways, and it is therefore seen that

$$(5.3) \quad \begin{aligned} c_{r,2n} &= 2 \sum_{k=r}^{n-1} \binom{k}{r} 2^{-(k+1)} z_{k,2n-2} \\ &= \frac{1}{2^{2n-2}} \sum_{k=r}^{n-1} \binom{k}{r} \binom{2n-2-k}{n-1}. \end{aligned}$$

A well-known formula for binomial coefficients<sup>3</sup> which can be proved by induction now shows that (5.2) is true.

In (5.2) we recognize the binomial distribution and we have the obvious.

**COROLLARY:**

$$(5.4) \quad c_{0,2n} > c_{1,2n} > c_{2,2n} > \dots > c_{n-1,2n}.$$

## 6. THE EXPECTATIONS.

**THEOREM 4.** *Let  $Z_{2n}$  and  $C_{2n}$  denote, respectively, the number of zeros and the number of changes of sign among the terms  $S_1, \dots,$*

<sup>3</sup> See, for example, formula (9.11) of Chapter 2 of the book quoted above.

$S_{2n-1}$ . For the expectations of these random variables we have

$$(6.1) \quad 2E(C_{2n}) = E(Z_{2n}) = 2nu_{2n} - 1$$

and thus

$$(6.2) \quad 2E(C_{2n}) = E(Z_{2n}) \sim \left(\frac{2}{\pi}\right)^{1/2} (2n)^{1/2} \quad \text{as } n \rightarrow \infty.$$

(These formulas shows that the density of the zeros and of changes of sign decreases at a fast rate.)

*Proof.* Define new random variables by  $Y_j = 1$  if  $X_j = 0$ , and  $Y_j = 0$  if  $X_j \neq 0$ . Then

$$(6.3) \quad 2E(C_{2n}) = E(Z_{2n}) = \sum_{j=1}^{2n-1} E(Y_j) = \sum_{r=1}^{n-1} u_{2r}.$$

and (6.1) follows by induction.

## 7. LATER RETURNS TO THE ORIGIN.

As a further application of the present elementary approach let us prove an important formula half of which has been proved by rather involved analytical methods <sup>4</sup>.

**THEOREM 5.** Let  $f_{k,2n}$  denote the probability that the  $k$ -th return to the origin takes place at the  $2n$ -th step (that is,  $f_{k,2n}$  is the probability that  $S_{2n} = 0$  and exactly  $r - 1$  among the  $S_j$  with  $1 \leq j < 2n$  vanish). Then

$$(7.1) \quad f_{k,2n} = z_{k,2n} - z_{k+1,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n} \frac{k}{2n-k}.$$

*Proof.* It is clear that  $f_{1,2n} = f_{2n}$  and that the  $f_{k,2n}$  satisfy the recurrence relation (4.9) and hence also (4.11). If we define  $f_{0,2n} = 0$  for  $n \neq 1$  and  $f_{0,0} = 1$ , then (7.1) is true for  $k = 0, 1$  and therefore for all  $k \geq 0$ .

Reçu le 17 mai 1957.

---

<sup>4</sup> See, for example, *ibid.*, formula (6.15) of Chapter 12.