

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 2 (1956)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** SOME PROBLEMS ON FINITE REFLECTION GROUPS  
**Autor:** Shephard, G. C.  
**Kapitel:** § 1. Introduction.  
**DOI:** <https://doi.org/10.5169/seals-32890>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 21.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# SOME PROBLEMS ON FINITE REFLECTION GROUPS

by

G. C. SHEPHARD, Birmingham

---

## § 1. INTRODUCTION.

Let  $V^r$  be an  $r$  dimensional vector space over some field  $k$  of zero characteristic. By a *hyperplane* we mean a linear subspace of  $V^r$  of  $r - 1$  dimensions. A linear transformation on  $V^r$  that is not the identity is called a *reflection* if it leaves some hyperplane pointwise invariant and is of finite order. If  $k = \mathbb{R}$ , the real numbers, then every reflection must be of order 2. Let  $W$  be a finite group of linear transformations on  $V^r$  such that the elements of  $W$  which are reflections generate  $W$ . Then  $W$  is said to be a *finite  $r$  dimensional group generated by reflections* or, more briefly, a *reflection group*.

The purely geometrical properties of reflection groups over  $\mathbb{R}$  have been discussed at length by H. S. M. COXETER and other authors (see the bibliography of [5]) and some of these have been extended by the author [8] to the case  $k = \mathbb{C}$ , the complex numbers. The irreducible reflection groups over  $\mathbb{C}$  have been enumerated, the complete list is given in [9; p. 301].

In this note we are primarily concerned with the algebraic properties of reflection groups. The first theorem due to Chevalley (see (i) of §3) states that every polynomial that is invariant under the transformations of a reflection group can be expressed as a polynomial in a set of  $r$  *basic invariant forms*  $I_1, I_2, \dots, I_r$ . Writing  $m_i + 1$  for the degree of  $I_i$ , the  $r$  integers  $m_i$  are called the *exponents* of the group  $W$ . The remaining theorems of § 3 are, in effect, properties of these integers. Several of these were first noticed by COXETER [6] for  $k = \mathbb{R}$  and further ones (together with the extension of COXETER's results to the complex numbers) are due to J. A. TODD and the author [9]. More recently the work of BOTT [2] has given a new interpretation ((iv) of § 3) to the exponents of a *crystallographic* group of reflections over  $\mathbb{R}$ , connecting them with the diagram of the corresponding Lie Group.

In stating the theorems in § 3, we give explicitly the restrictions that must be placed on the ground field  $k$  and also on the group  $W$ . For each theorem it is briefly indicated how the result may be proved. Sometimes a proof in general terms is known, but in the majority of cases it has been necessary to verify the properties one by one for all the irreducible groups over  $C$  and then show that (with the exception of (vi)) they extend to the reducible groups. These two distinct methods will be referred to as *proving* and *verifying* respectively. Perhaps the most remarkable fact is that the result (iv) of § 3 which appears to concern itself entirely with discrete infinite groups generated by reflections has been proved only by topological methods (spectral sequences and Morse theory). A direct proof, avoiding the topology, would be interesting. Further outstanding problems are the discovery of proofs for those properties that have so far only been verified, and the extension of these theorems to more general fields  $k$ , especially to the case where  $k$  is of finite characteristic.

## § 2. THE CONNECTION BETWEEN LIE GROUPS AND REFLECTION GROUPS.

Let  $G$  be an  $n$  dimensional compact semi-simple Lie Group. A maximal connected abelian subgroup of  $G$  forms a submanifold of  $G$  which is a torus of dimension  $r$  (the *rank* of  $G$ ) [10]. This is called a *maximal torus*  $T$  of  $G$ . The inner automorphisms of  $G$  by elements of  $N_T$ , the normaliser of  $T$ , induce a finite group of automorphisms of  $T$ . These in turn induce linear transformations of the tangent space  $V^r$  to  $T$  at the identity  $e$ . It can be shown that this group of linear transformations forms a reflection group over  $R$  called the *Weyl group*  $W$  of  $G$ . This group has the further property that it is *crystallographic*, i.e. by suitable choice of coordinates it is represented by a set of matrices whose coefficients are integers, or, alternatively, if the coordinates are chosen so that the matrices are orthogonal (so that  $W$  is then a group of congruent transformations acting on a Euclidean space  $R^n$ ) then the angle between any two hyperplanes of reflection of  $W$  is an integral multiple of  $\pi/4$  or  $\pi/6$ .